Magooey's Math Problems

Mean Value Theorems

Synopsis. Mean Value Theorems deal with functions that are continuous on a closed interval $[a, b]$ but differentiable in the open interval (a, b). To develop these Theorems, we need some results about continuous functions. In particular, we have the following.

Theorem (Continuity Property). *Consider a continuous function f on a closed and bounded interval $[a, b]$. Then the range of f over all x in $[a, b]$ is a closed and bounded interval $[c, d]$.*

This immediately implies what may be called the Extreme Value Theorem.

Theorem (Extreme Value Theorem). *A continuous function on a closed and bounded interval attains its maximum and minimum over that interval.*

Proof. This follows immediately from the continuity property. For f continuous on $[a, b]$, the Continuity Property tells us that the range is some closed and bounded interval $[c, d]$. Thus the minimum of f over $[a, b]$ is c, and the maximum is d. Since both c and d are in the image of f, there exists values x_c and x_d from the interval $[a, b]$ for which $f(x_c) = c$ and $f(x_d) = d$. Thus the maximum and minimum values are attained by f. ∎

A typical non-example is the function $f(x) = 1/x$ defined on the open interval $(1/2, 1)$. Then the image of f is the open interval $(1, 2)$, and no x in the domain of f yields either the value 1 or 2.

The proof of the Continuity Property requires some advanced concepts that will take us too far afield at this point. An Appendix with a proof may be available at a later juncture.

As an example of the usefulness of these theorems, consider the function $f(x) = x^3 - 3x + 1$. We claim that this function has a root in the interval $[0, 1]$. Note that f is continuous, and $f(0) = 1$ while $f(1) = -1$. By the Continuity Property, the range of f is a closed interval that includes $[-1, 1]$. In particular, there is some value of x in the domain $[0, 1]$ for which $f(x) = 0$ as 0 belongs to the set $[-1, 1]$.

Our first instance of a Mean Value Theorem is Rolle's Theorem.

Theorem (Rolle's Theorem). *Let f be a function continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Suppose also that $f(a) = f(b) = c$, for some constant c. Then there exists an x_0 in $[a, b]$ for which $f'(x_0) = 0$.*

Proof. Since f is continuous on the closed interval $[a, b]$, it attains it maximum value H and its minimum value l over this interval. Suppose either the maximum or minimum value is obtained at some point x_0 in the open interval (a, b). Then x_0 is a local maximum or a local minimum, and f is differentiable at c. This implies $f'(x_0) = 0$ and we are done.

The other possibility is that the maximum and minimum values for f are attained at the endpoints a and b. But $f(a) = f(b) = c$ and this forces the function f to be constant and equal to c over the whole interval $[a, b]$. Then $f' = 0$ over the whole open interval (a, b) and any point chosen in this set will do. ∎

From Rolle's Theorem, we can get extract the Mean Value Theorem.

Theorem (Mean Value Theorem). *Suppose f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then for some x_0 in (a, b) we have*

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}.$$

Proof. The approach is to modify $f(x)$ by adding a constant time x, such that the new function satisfies the conditions of Rolle's Theorem. Let $F(x) = f(x) + h\,x$ where h is a currently unknown constant. We wish to have $F(a) = F(b)$. It follows that h must satisfy

$$\begin{aligned} F(a) &= F(b) \\ f(a) + h\,a &= f(b) + h\,b \\ h\,(a - b) &= f(b) - f(a) \\ h &= -\frac{f(b) - f(a)}{b - a}. \end{aligned}$$

With this value of h we can apply Rolle's Theorem, so there is some x_0 in the open interval (a, b) such that $F'(x_0) = 0$. That becomes $f'(x_0) + h = 0$ or equivalently $f'(x_0) = -h = \dfrac{f(b) - f(a)}{b - a}$. ∎

The Mean Value Theorem has nontrivial consequences. Here is one.

Theorem. *Suppose $f(x)$ is a continuous function on $[a, b]$ and differentiable on (a, b). If $f'(x) = 0$ for all x in (a, b) then f is constant on the closed interval $[a, b]$.*

Proof. Choose any y with $a < y \le b$. Then $f(x)$ satisfies the Mean Value Theorem on the interval $[a, y]$. Therefore there exists an x_0 in the open interval (a, y) for which

$$f'(x_0) = \frac{f(y) - f(a)}{y - a}$$

Since $f'(x_0) = 0$ we get $f(y) = f(a)$ for all y with $0 < y \le b$. That means f is constant on $[a, b]$. ■

Another consequence of the above results is the Cauchy Mean Value Theorem.

Theorem (Cauchy Mean Value Theorem). *Let $f(x)$ and $g(x)$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose also that $g'(x)$ is never zero on (a, b). Then there exists x_0 in (a, b) for which*

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof. Set $F(x) = f(x) + h\,g(x)$ where h is a constant to be determined. We wish to choose h so that Rolle's Theorem can be applied. Thus we set $F(a) = F(b)$ and solve for h.

$$\begin{aligned} f(a) + h\,g(a) &= f(b) + h\,g(b) \\ h\,(g(a) - g(b)) &= f(b) - f(a) \\ h &= -\frac{f(b) - f(a)}{g(b) - g(a)}. \end{aligned}$$

By Rolle's Theorem, there is an x_0 in (a, b) for which $F'(x_0) = 0$. This implies $f'(x_0) + h\,g'(x_0) = 0$. Since g' is never zero on (a, b) it follows that

$$\frac{f'(x_0)}{g'(x_0)} = -h = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

■

A consequence of the Cauchy Mean Value Theorem is the technique for finding limits of certain ratios, called l'Hôpital's Rule.

Corollary (l'Hôpital's Rule). *Let $f(x)$ and $g(x)$ satisfy the conditions of the Cauchy Mean Value Theorem. Additionally suppose we have $f(a) = g(a) = 0$. Then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided the limits exist.

Proof. We have $b \neq a$, and an x_0 between a and b such that

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f(b)}{g(b)}.$$

Therefore as $b \to a$ it follows that $x_0 \to a$ so

$$\lim_{x_0 \to a} \frac{f'(x_0)}{g'(x_0)} = \lim_{b \to a} \frac{f(b)}{g(b)}$$

from which it immediately follows that

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f(x)}{g(x)}$$

provided the limits exist. ■

As an example, let us calculate $\lim_{x \to 0} \frac{\sin(17\,x)}{4\,x}$. Since the numerator and denominator equal 0 at $x = 0$, we simply cannot just evaluate the functions at zero and obtain the limit. We apply l'Hôpital's Rule in the hope that something simpler will show up when we differentiate each of the numerator and the denominator.

$$\lim_{x \to 0} \frac{\sin(17\,x)}{4\,x} = \lim_{x \to 0} \frac{\frac{d}{dx}(\sin(17\,x))}{\frac{d}{dx}(4\,x)} = \lim_{x \to 0} \frac{17\,\cos(17\,x)}{4} = \frac{17}{4}.$$

As another example, let us calculate $\lim_{x \to 0} \frac{1 - \cos(x)}{x\,\sin(x)}$. Again the numerator and denominator equal 0 at $x = 0$. We need to apply l'Hôpital's Rule twice, as follows.

$$\begin{aligned}
\lim_{x \to 0} \frac{1 - \cos(x)}{x\,\sin(x)} &= \lim_{x \to 0} \frac{\frac{d}{dx}(1 - \cos(x))}{\frac{d}{dx}(x\,\sin(x))} \\
&= \lim_{x \to 0} \frac{\sin(x)}{x\cos(x) + \sin(x)} \\
&= \lim_{x \to 0} \frac{\frac{d}{dx}\sin(x)}{\frac{d}{dx}(x\cos(x) + \sin(x))} \\
&= \lim_{x \to 0} \frac{\cos(x)}{-x\,\sin(x) + \cos(x) + \cos(x)} \\
&= \frac{1}{2}.
\end{aligned}$$

Exercises.

1. Find a value of x_0 which verifies Rolle's Theorem when $f(x) = x^3 - x$ on the interval $[0,\ 1]$.

Solution. Note that $f(0) = f(1) = 0$, so Rolle's Theorem applies. We need to find some x_0 in $[0,\ 1]$ such that $f'(x_0) = 0$. That is, we must solve $3\,x_0^2 - 1 = 0$. The possible solutions are $x_0 = \pm\sqrt{3}/3$. Since the interval in question is $[0,\ 1]$ the answer is $x_0 = \sqrt{3}/3$. ■

2. Find a value of x_0 which verifies Rolle's Theorem for $f(x) = x + \dfrac{1}{x}$ on the interval $[1/2,\ 2]$.

3. Find the quarter of the interval $[0,\ 1]$ where $f(x) = x^3 - 3\,x + 1$ must have a root.

Solution. We note that $f(0) = 1$ and $f(1) = -1$, so the interval $[0,\ 1]$ does in fact contain a root of f. Compute $f(1/2) = -3/8$. It follows that the interval $[0,\ 1/2]$ must contain a root of f. We now compute $f(1/4) = 17/64$. Hence the interval $[1/4,\ 1/2]$ must contain a root of f, and is the required quarter interval. ■

4. Find a quarter of the interval $[1,\ 2]$ where $f(x) = 2\,x^2 - \dfrac{5}{x}$ must have a root.

5. Find $\displaystyle\lim_{x\to 0} \frac{1-\cos(x)}{x\ \sin(x)}$ by using our knowledge of the limits of $\sin(x)/x$ and $(1-\cos(x))/x^2$ as $x \to 0$, instead of l'Hôpital's Rule.

6. Find a quarter of the interval $[0, \pi/2]$ where the function $f(x) = x\ \sin(x) - 1$ must have a root.

Solution. We start with $f(0) = -1$ and $f(\pi/2) = (\pi/2)\cdot 1 - 1 = \pi/2 - 1 \approx 0.5707$. So the interval $[0,\ \pi/2]$ certainly contains a root. We should now evaluate the function f at $\pi/4$. We find $f(\pi/4) \approx -0.4446$. Hence there must be a root in the interval $[\pi/4,\ \pi/2]$. Computing, $f(3\,\pi/8) \approx 0.0884$. Therefore the quarter of the original interval that must have a root is $[\pi/4,\ 3\,\pi/8]$. ■

7. Find a value of x_0 which verifies the Mean Value Theorem when $f(x) = x^3 + 2\,x^2 - 5$ on the interval $[1,\ 2]$.

Solution. We first compute $\dfrac{f(b)-f(a)}{b-a} = \dfrac{f(2)-f(1)}{2-1}$ which simplifies to 13. Since $f'(x) = 3\,x^2 + 4\,x$ we wish to solve $3\,x^2 + 4\,x = 13$. By the quadratic formula, $x_0 = \dfrac{-2\pm\sqrt{43}}{3}$. Since we are constrained to the interval $[1,\ 2]$ it follows that $x_0 = \dfrac{-2+\sqrt{43}}{3}$. ■

8. Find a value of x_0 which verifies the Cauchy Mean Value Theorem for $f(x) = x^3 + 3\,x^2 - 2\,x$ and $g(x) = 2\,x^2 + x$ on the interval $[-1,\ 1]$.

9. Find the limit. $\displaystyle\lim_{x\to 0} \frac{11\,x}{\sin(7\,x)}$

10. Find the limit. $\lim_{x\to 0} \dfrac{\sin(x^2)}{x \sin x}$

<u>*Solution*</u>. This is a case where we have to use l'Hôpital's Rule twice.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(x^2)}{x \sin(x)} &= \lim_{x\to 0} \frac{2\,x \cos(x^2)}{x \cos(x) + \sin(x)} \\
&= \lim_{x\to 0} \frac{(2\,x)(2\,x\,(-\sin(x^2))) + 2 \cos(x^2)}{x\,(-\sin(x)) + \cos(x) + \cos(x)} \\
&= \lim_{x\to 0} \frac{-4\,x^2 \sin(x^2) + 2 \cos(x^2)}{-x \sin(x) + 2 \cos(x)} \\
&= \frac{2}{2} = 1\,.
\end{aligned}$$

■

11. Find the limit. $\lim_{x\to 0} \dfrac{(1+x)^{1/3} - 1}{x}$

<u>*Solution*</u>.

$$\begin{aligned}
\lim_{x\to 0} \frac{(1+x)^{1/3} - 1}{x} &= \lim_{x\to 0} \frac{\frac{d}{dx}\left((1+x)^{1/3} - 1\right)}{\frac{d}{dx}(x)} \\
&= \lim_{x\to 0} \frac{(1/3)\,(1+x)^{-2/3}}{1} = \frac{1}{3}\,.
\end{aligned}$$

■

12. Find the limit. $\lim_{x\to 0} \dfrac{(1+2\,x)^{1/5} - 1}{x}$