# Magooey's Math Problems



## Maxima and Minima

**Synopsis.** This section covers calculus problems for maxima and minima. A local maxima or a local minima for a function $f(x)$ is a real value $a$ for which $f(a)$ is a maximum value (respectively minimum value) for some open interval around $a$. Recall that local maxima and local minima of a function $f(x)$ which is differentiable on an open interval must occur at points where the derivative $f'(a)$ equals $0$. A value, $x$ where $f'(a) = 0$ is called a stationary point. Not all stationary points are local maxima or local minima.

We define the critical points of the function $f(x)$ on an interval to be any of the stationary points, point where the derivative $f'(x)$ does not exist, or an endpoint of the interval. To find maxima and minima for functions $f(x)$ on an interval, we need only check the critical points and compute where the maximum and minimum values of $f$ occur.

### Exercises.

**1.** Find the maximum value of $f(x) = -3\,x^2 + 12\,x + 5$ where $x$ can vary over the real line.

*Solution*. We first find the derivative $f'(x)$.

$$f'(x) \;=\; \frac{d}{dx}\left(-3\,x^2 + 12\,x + 5\right) \;=\; (-3)\,2x + 12 \;=\; -6x + 12\,.$$

Then we find the critical values of $x$. But that just means solving $f'(x) = 0$. We immediately find $x = 2$ as the solution. Plugging this in we get $f(2) = -3 * 2^2 + 12 * 2 + 5 = -12 + 24 + 5 = 17$.

We also must check the endpoints of the interval under consideration. But as $x \longrightarrow \pm\infty$ $f(x)$ goes to $-\infty$ due to the dominating term $-3\,x^2$. Hence the maximum value of $F(x)$ on the real line is at $x = 2$, thus 17. ■

**2.** Find the minimum value of $f(x) = x + \dfrac{1}{9\,x}$ where $x > 0$.

*Solution I*. We first find $f'(x)$.

$$f'(x) \;=\; \frac{d}{dx}\left(x + \frac{1}{9\,x}\right) \;=\; 1 + \frac{d}{dx}\left(\frac{1}{9}\,x^{-1}\right) \;=\; 1 - \frac{1}{9}x^{-2}\,.$$

Solving for $f' = 0$ we find $1 = \frac{1}{9}x^{-2}$ or $9 = x^{-2}$ or $x = \pm\frac{1}{3}$. Since we are restricted to $x > 0$ we may ignore the negative solution. At $x = \frac{1}{3}$ we find the value

$$f(x) \;=\; \frac{1}{3} + \frac{1}{9 \cdot \frac{1}{3}} \;=\; \frac{1}{3} + \frac{1}{3} \;=\; \frac{2}{3}\,.$$

Now we note that as $x \longrightarrow \infty$, $f(x) \longrightarrow +\infty$, while when $x$ goes to 0 from above $\frac{1}{9\,x}$ also goes to $+\infty$. Hence the minimum value of the function $f(x)$ in the region $x > 0$ is $f\left(\frac{1}{3}\right) = \frac{2}{3}$. ∎

*Solution II*. We can actually solve this problem by an elementary method, using the Theorem on Arithmetic Means and Geometric Means. In the case of two nonnegative variables $a$, $b$, this Theorem states that

$$\sqrt{a\,b} \;\le\; \frac{a+b}{2}\,.$$

Furthermore, equality holds only in the case $a = b$. This is easy to prove directly by algebra. For the above problem, we set $a = x$ and $b = \frac{1}{9\,x}$. We find

$$\sqrt{x \cdot \frac{1}{9\,x}} \;\le\; \frac{x + \frac{1}{9\,x}}{2}$$

or

$$\frac{1}{3} \;=\; \sqrt{\frac{1}{9}} \;\le\; \frac{x + \frac{1}{9\,x}}{2}$$

while gives us

$$\frac{2}{3} \;\le\; x + \frac{1}{9\,x}$$

with equality only when $x = \frac{1}{9\,x}$, or when $x = \frac{1}{3}$ when we restrict to $x > 0$. ∎

**3.** Find the minimum value of $f(x) = x + \frac{1}{16\,x^2}$ where $x > 0$.

**4.** A rectangle has vertices with coordinates $\{(0,\,0),\,(a,\,0),\,(0,\,b),\,(a,\,b)\}$. If the vertex $(a,\,b)$ lies on the portion of the ellipse $\frac{x^2}{4} + \frac{y^2}{25} = 1$ that is in the first quadrant, find the maximum possible area of the rectangle.

**5.** Same setup as the previous problem, but the vertex $(a, b)$ lies on the curve $\sqrt{x}+\sqrt{y} \leq 6$.

**6.** A soup can, in the shape of a closed cylinder, is made of $8\pi$ square inches of tin. Find the maximum value of the volume that can be contained in the soup can.

**7.** A cylinder of radius $r$ and height $h$ is topped by a hemisphere. If the surface area of the outside of this figure is $40\pi$, what is the maximum volume it can contain? (Hint: Recall that the surface area of a sphere is $4\pi r^2$ while its volume is $\frac{4}{3}\pi r^3$)

**8.** Find the minimum distance between the point $(2, 5)$ and the curve $2x + y = 3$.

**9.** Find the maximum and minimum value of the function $f(x, y) = x^2 + xy - y^2$ over the rectangular region $-\frac{1}{2} \leq x, y \leq \frac{1}{2}$. (Hint: Fix $x$ and find the extreme values for each $x$. Then take the maximum and minimum as $x$ varies over the interval $[-\frac{1}{2}, \frac{1}{2}]$).

**10.** Prove the Theorem on Arithmetic Means and Geometric Means by Calculus. This Theorem states that for all nonnegative $x_1, x_2, \ldots, x_n$ the following inequality holds.

$$\sqrt[n]{x_1 \cdot x_2 \cdot \cdots \cdot x_n} \;\leq\; \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

The left side of the equation is called the *Geometric Mean* of $x_1, x_2, \ldots, x_n$ while the right side is called the *Arithmetic Mean* (or just the mean) of the $x$'s. (Hint: The case of $n = 2$ is a simple inequality to prove. Then proceed by induction on $n$, working with a properly chosen function $f_n(x)$.)

*Solution*. For the case $n = 2$ we wish to prove $\sqrt{a\,b} \leq \frac{a+b}{2}$. This is equivalent to $0 \leq \frac{a - 2\sqrt{a\,b} + b}{2}$ which can be written $0 \leq \frac{(\sqrt{a} - \sqrt{b})^2}{2}$ which is always true. We note that the inequality is strict, i.e., $\sqrt{a\,b} < \frac{a+b}{2}$ unless $a = b$.

Now assume the Theorem is true for the case of $n$ nonnegative variables, namely $x_1, x_2, \ldots, x_n$. We wish to prove it true in the case of $n+1$ nonnegative variables, which we will call $x_1, x_2, \ldots, x_n, t$. In other words, we can assume by induction that

$$\sqrt[n]{x_1 \cdot x_2 \cdot \cdots \cdot x_n} \;\leq\; \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

for any nonnegative $x_1, x_2, \ldots, x_n$, and we wish to show that

$$\sqrt[n+1]{x_1 \cdot x_2 \cdot \cdots \cdot x_n\, t} \;\leq\; \frac{x_1 + x_2 + \cdots + x_n + t}{n+1}.$$

for any $t \geq 0$. Set

$$f(t) \;=\; \frac{x_1 + x_2 + \cdots + x_n + t}{n+1} \;-\; (x_1 \cdot x_2 \cdot \cdots \cdot x_n \cdot t)^{\frac{1}{n+1}}.$$

Then $f(0) \geq 0$ and $f(t) \longrightarrow \infty$ as $t \longrightarrow \infty$. We compute $f'(t)$. Set $c = x_1 \cdot x_2 \cdot \cdots \cdot x_n$. Then c is a constant.

$$f'(t) \;=\; \frac{1}{n+1} - \frac{1}{n+1}\, c^{\frac{1}{n+1}}\, t^{-\frac{n}{n+1}}\,.$$

Now $f'$ is continuous on the open interval $(0, \infty)$ and is not defined at $t = 0$. Thus we must include $t = 0$ when determining maxima and minima for $f$ on the interval $x \geq 0$. Solving for $f' = 0$ we find

$$\begin{aligned} 1 &= c^{\frac{1}{n+1}}\, t^{-\frac{n}{n+1}} \\ t^n &= c \\ t &= c^{\frac{1}{n}} \;=\; (x_1 \cdot \cdots \cdot x_n)^{\frac{1}{n}}\,. \end{aligned}$$

But

$$\begin{aligned} f\left(c^{\frac{1}{n}}\right) &= \frac{x_1 + \cdots\cdots x_n + c^{\frac{1}{n}}}{n+1} \;-\; \left(c \cdot c^{\frac{1}{n}}\right)^{\frac{1}{n+1}} \\ &= \frac{x_1 + \cdots + x_n + c^{\frac{1}{n}}}{n+1} \;-\; c^{\frac{1}{n}} \end{aligned}$$

and we are given by induction that

$$\frac{x_1 + \cdots + x_n}{n} \;\geq\; c^{\frac{1}{n}} \quad \text{or} \quad x_1 + \cdots + x_n \;\geq\; n \cdot c^{\frac{1}{n}}\,.$$

Substituting, we find

$$f\left(c^{\frac{1}{n}}\right) \;\geq\; \frac{nc^{\frac{1}{n}} + c^{\frac{1}{n}}}{n+1} \;-\; c^{\frac{1}{n}} \;=\; 0\,.$$

Since $f \geq 0$ at all the extremal points, namely zero and $c^{\frac{1}{n}}$, and since $f \longrightarrow \infty$ as $x \longrightarrow \infty$, we can conclude that $f(t) \geq 0$ for all $t \geq 0$. This is the statement of the Theorem on Arithmetic Means and Geometric Means for the case of $n+1$. ■

**11.** Prove that equality holds in the Theorem on the Arithmetic Mean and Geometric Mean only in the case $x_1 = x_2 = \cdots = x_n$.