Magooey's Math Problems

Limits of Trigonometric Functions

Synopsis. The trigonometric functions, sine, cosine, tangent and their inverses, have a number of properties that arise in Calculus and its applications. The periodic nature of sine and cosine becomes useful in modeling the behavior of a swinging pendulum or even economic activity. A few key limits will unlock many of the properties of the trigonometric functions.

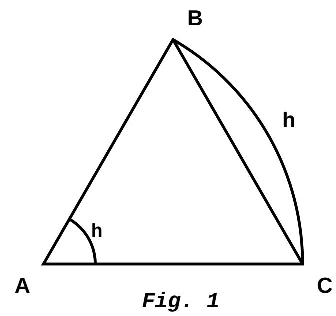

Fig. 1

Firstly we wish to prove that the trigonometric functions are continuous, except where their values go to $\pm\infty$. For example tangent can not possibly be continuous at $\pi/2$, but is continuous elsewhere in the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. In order to obtain the results on continuity, we first prove the following two formulas.

$$\lim_{x\to 0} \sin(x) = 0\,, \qquad \lim_{x\to 0} \cos(x) = 1\,.$$

For the first limit, we consider the sector of angle h in a circle of radius 1. The area of this sector can be gotten by the ratio of the angle h to the whole circle, which corresponds to an angle of 2π. Thus

$$\frac{\text{area of sector}}{h} = \frac{\pi \cdot 1^2}{2\pi}$$

which gives the area of the sector as $h/2$. This area must be greater than that of the triangle $\triangle ABC$ inscribed in the sector. The triangle has two sides of length 1 and an included angle of h. Therefore the area of the triangle is $\frac{1}{2} \cdot 1 \cdot 1 \cdot \sin(h)$. So we find

$$\frac{\sin(h)}{2} \le \frac{h}{2} \quad \text{or} \quad \sin(h) \le h\,.$$

This holds for all $h \ge 0$. In particular we note that $\sin(0) = 0$. For $h < 0$, $|\sin(h)| = -\sin(h) = \sin(-h) = \sin(|h|) \le |h|$. For $h > 0$ the previous inequality also implies $|\sin(h)| = \sin(h) \le h = |h|$.

To show the first limit, we need to show for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - 0| < \delta$ implies $|\sin(x) - \sin(0)| < \varepsilon$. Since $|\sin(x)| \le |x|$ for all x, we merely choose $\delta = \varepsilon$, and the limit is proven.

For the second limit, use the Law of Cosines in the inscribed triangle. Let the side opposite the angle h have length a. Then the Law of Cosines says $a^2 = 1^2 + 1^2 - 2 \cdot 1 \cdot 1 \cos(h)$ or $a^2 = 2 - 2\cos(h)$. Since a line between two points is the shortest distance between those points, a is less than the part of the sector that lies on the circumference of the circle. The circumference of the sector of angle h can be found by proportion to the circumference of the whole circle.

$$\frac{\text{circumference of sector}}{h} = \frac{\text{circumference of circle}}{\text{radian measure of circle}} = \frac{2 \cdot \pi \cdot 1}{2\pi}.$$

Thus the circumference of the sector is simply h. From $a \leq h$ we get $a^2 \leq h^2$ or $2 - 2\cos(h) \leq h^2$.

Equivalently, $1 - \cos h \leq h^2/2$. Since $\cos(-h) = \cos(h)$ we have for all h, $1 - \cos(h) \leq h^2/2$. Using the Sandwich Theorem, with the functions $x^2/2$ and 0, we find $\lim_{x \to 0}(1 - \cos(x)) = 0$. We can now prove the continuity of the sine and cosine functions.

Theorem. *The function* $\sin(x)$ *is continuous for all real* x.

Proof. We start with the formulas

$$\begin{aligned} \sin(A+B) &= \sin(A)\cos(B) + \cos(A)\sin(B) \\ \sin(A-B) &= \sin(A)\cos(B) - \cos(A)\sin(B). \end{aligned}$$

Subtracting, we get

$$\sin(A+B) - \sin(A-B) = 2\cos(A)\sin(B).$$

We wish to show $\sin(x)$ has the limit $\sin(a)$ as $x \to a$. Set $A - B = a$ and $A + B = x$. Then $A = (x+a)/2$ and $B = (x-a)/2$. The formula becomes

$$\sin(x) - \sin(a) = 2\cos\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right).$$

We know from above that $\left|\sin\left(\frac{x-a}{2}\right)\right| \leq \frac{|x-a|}{2}$ while cosine of any angle is at most 1 in absolute value. Therefore

$$|\sin(x) - \sin(a)| \leq 2 \cdot 1 \cdot \frac{|x-a|}{2} = |x-a|.$$

Therefore given any $\varepsilon > 0$ choose $\delta = \varepsilon$. Then $|x-a| < \delta$ forces $|\sin(x) - \sin(a)| \leq |x-a| < \delta = \varepsilon$. That completes the proof of continuity. ■

The cosine function can be proved continuous similarly. This is left as an exercise.

Next we need the more delicate limit results.

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1\,, \qquad \lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}\,.$$

These will be used in computing the derivatives of the sine and cosine functions. Returning to the first diagram, we already have $\sin(h) < h$ for all angles $h > 0$. Draw a tangent to the circle from vertex B of the triangle (see Figure 2). This tangent touches line $\overline{AC}$ at some point D outside the circle and makes a right triangle with A as the third vertex. The area of this triangle is greater than the area of the sector of the circle of angle h at the center. It is clear that the length of the side of this triangle opposite angle h is $\tan(h)$ as the other side is equal to the radius, 1, in length.

By comparing areas we get the inequality $h \le \tan(h) = \dfrac{\sin(h)}{\cos(h)}$. Putting both inequalities together we find for $h > 0$

$$\cos(h) \;\le\; \frac{\sin(h)}{h} \;\le\; 1\,.$$

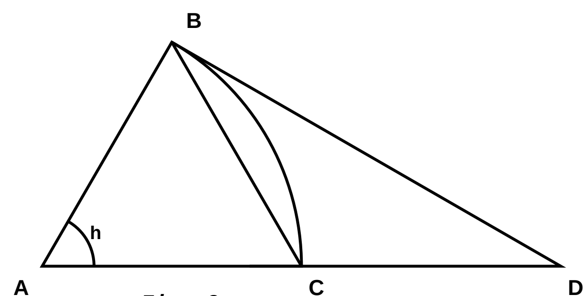

Fig. 2

For $h < 0$, note that $\dfrac{\sin(h)}{h} = \dfrac{-\sin(h)}{-h} = \dfrac{\sin(-h)}{-h}$ which is constrained between 1 and $\cos(-h) = \cos(h)$. Therefore, the above inequality holds for all $h \ne 0$. By the Sandwich Theorem, we can conclude that $\lim_{h\to 0} \dfrac{\sin(h)}{h} = 1$.

Finally we consider the limit as $h \to 0$ of $\dfrac{1-\cos^2(h)}{h^2}$. By the product rule for limits

$$\begin{aligned} 1 &= \lim_{h\to 0} \frac{\sin(h)}{h} \cdot \lim_{h\to 0} \frac{\sin(h)}{h} = \lim_{h\to 0} \frac{\sin^2(h)}{h^2} \\ &= \lim_{h\to 0} \frac{1-\cos^2(h)}{h^2} = \lim_{h\to 0} \frac{(1-\cos(h))\,(1+\cos(h))}{h^2} \\ &= \lim_{h\to 0} (1+\cos(h)) \cdot \lim_{h\to 0} \frac{1-\cos(h)}{h^2} = 2 \lim_{h\to 0} \frac{1-\cos(h)}{h^2}\,. \end{aligned}$$

From this we conclude that $\lim_{h\to 0} \dfrac{1-\cos(h)}{h^2} = \dfrac{1}{2}$.

It is interesting to note that we can obtain the limit $\lim_{h\to 0} \frac{1-\cos^2(h)}{h^2}$ without recourse to the result for the limit as $x \to 0$ of $\sin(x)/x$. Consider Figure 3, which is based on Figure 1. We draw the angle bisector of $\angle A$ which touches the circle at point G and intersects the chord BC at E. By symmetry, the angles $\angle AEB$ and $\angle AEC$ are 90^o, so the respective triangles are right triangles.

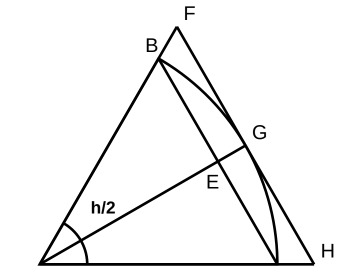

Fig. 3

Next we draw a tangent at the point G and extend it to intersect each of the lines AB and AC. The tangent intersects the line AB at the point F, and it intersects the line AC at the point H. We know $\angle AGF$ and $\angle AGH$ are 90^0 angles. Thus the triangles $\triangle AEB$ and $\triangle AGF$ are similar. By symmetry we find that GF equals GH in length. It follows that

$$\frac{AE}{AG} = \frac{BE}{FG} = \frac{BC}{FH}$$

Since this is a unit circle, we have the length of AG equal to 1. Also the length of AE is $\cos(h/2)$ since the length of AB is equal to 1. We have previously calculated the length of BC to be $\sqrt{2}\sqrt{1-\cos(h)}$. Substituting, we find

$$\frac{\cos(h/2)}{1} = \frac{\sqrt{2}\sqrt{1-\cos(h)}}{FH}$$
$$FH = \frac{\sqrt{2}\sqrt{1-\cos(h)}}{\cos(h/2)}.$$

Since the area of $\triangle AFH$ is greater than the area of the sector of the circle of angle h, we know that one-half the length of FH times the height of the perpendicular dropped from A to FH is greater than the quantity $h/2$. But the perpendicular in this case is AG which is of length 1. It follows that

$$\frac{1}{2}FH > \frac{h}{2}$$
$$\frac{\sqrt{2}\sqrt{1-\cos(h)}}{\cos(h/2)} > h.$$

Squaring and simplifying, we get

$$\frac{1-\cos(h)}{h^2} > \frac{\cos^2(h/2)}{2}.$$

Putting this together with the previous inequality regarding $1-\cos(h)$ we find that

$$\frac{1}{2} > \frac{1-\cos(h)}{h^2} > \frac{\cos^2(h/2)}{2}.$$

This inequality also holds for $h < 0$. Since $\lim_{h\to 0} \cos^2(h/2) = 1$ we can conclude by the Sandwich Principle that $\lim_{h\to 0} \frac{1-\cos(h)}{h^2} = \frac{1}{2}$.

Exercises.

1. Find the limit. $\lim_{x\to 0} \frac{\sin(17\,x)}{x}$

Solution.

$$\lim_{x\to 0} \frac{\sin(17\,x)}{x} = \lim_{x\to 0} \left\{ 17 \cdot \frac{\sin(17\,x)}{17\,x} \right\} = 17 \cdot \lim_{x\to 0} \frac{\sin(17\,x)}{17\,x}\,.$$

Set $u = 17\,x$. As $x \to 0$ it follows that u must also go to 0. The above limit is then equal to

$$17 \cdot \lim_{x\to 0} \frac{\sin(u)}{u} = 17 \cdot 1 = 17\,.$$

So the answer is 17. ■

2. Find the limit. $\lim_{x\to 0} \frac{\sin(5\,x)}{3\,x}$

Solution.

$$\lim_{x\to 0} \frac{\sin(5\,x)}{3\,x} = \frac{1}{3} \lim_{x\to 0} \frac{\sin(5\,x)}{x} = \frac{1}{3} \lim_{x\to 0} \left\{ 5\, \frac{\sin(5\,x)}{5\,x} \right\} = \frac{5}{3} \lim_{x\to 0} \frac{\sin(5\,x)}{5\,x}\,.$$

Let $u = 5\,x$. Then as $x \to 0$ it follows that u also goes to 0. The above equation is equal to

$$\frac{5}{3} \lim_{u\to 0} \frac{\sin(u)}{u} = \frac{5}{3} \cdot 1 = \frac{5}{3}\,.$$

So the answer is $5/3$. ■

3. Find the limit. $\lim_{x\to 0} \frac{\sin(7\,x)}{\sin(4\,x)}$

4. Find the limit. $\lim_{x\to 0} \frac{\sin(x)\,\cos(2\,x)}{2\,x}$

5. Find the limit. $\lim_{x\to 0} \frac{\sin(2\,x)}{\cos(2\,x)}$

6. Find the limit. $\lim_{x\to 0} \frac{1-\cos(x)}{x}$

Solution.

$$\lim_{x\to 0} \frac{1-\cos(x)}{x} = \lim_{x\to 0} \left\{ x\, \frac{1-\cos(x)}{x^2} \right\} = \lim_{x\to 0} x \cdot \lim_{x\to 0} \frac{1-\cos(x)}{x^2} = 0 \cdot \frac{1}{2} = 0\,.$$

So the answer is 0. ■

7. Find the limit. $\lim_{x \to 0} \dfrac{1 - \cos(2x)}{x^2}$

8. Show $\lim_{h \to 0} \cos^2(h/2) = 1$

Solution. We know that $\lim_{h \to 0} \cos(h) = 1$. Let $u = h/2$. Then as $h \to 0$ it follows that $u \to 0$. So $\lim_{h \to 0} \cos(h/2) = \lim_{u \to 0} \cos(u) = 1$. By the product rule for limits, it follows that $\lim_{h \to 0} \cos^2(h/2) = 1 \cdot 1 = 1$. ■

9. Show that the function $\cos(x)$ is continuous for all real x.

Solution. Probably the easiest way to demonstrate that $\cos(x)$ is continuous is to recall that $\sin(x)$ is continuous and that $\cos(x) = \sin\left(\frac{\pi}{2} - x\right)$. Then for any real a

$$\lim_{x \to a} \cos(x) \;=\; \lim_{x \to a} \sin\left(\frac{\pi}{2} - x\right) \;=\; \sin\left(\frac{\pi}{2} - a\right) \;=\; \cos(a)\,.$$

Thus cosine is continuous at a. ■

10. Once we know that $\sin(x)$ and $\cos(x)$ are continuous for all x, why does it follow that $\tan(x)$, $\cot(x)$, $\sec(x)$ and $\csc(x)$ are continuous except where they go to $\pm\infty$?