MAGOOEY'S MATH PROBLEMS

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Limits and Sequences

Synopsis. Calculus was developed in the late Renaissance and early Modern eras to deal with problems of motion, distance, area and volume. In working with these problems, some concepts were developed and formalized to enable a rigorous mathematical treatment. For example, you may wish to find the velocity of an object that hits the ground after being tossed from a height. Indeed it would be nice to find the velocity of the object at any instant, until it touches the ground. This could be estimated by taking the distance traveled over a small time interval, divided by the time interval. This would give the average velocity over the time interval. To get the instantaneous velocity, we would like to take the time interval to be shorter and shorter. If the average velocity approaches a particular number, the shorter the time interval is taken, you may then wish to say it approaches a "limit".

We define a sequence of real numbers $\langle a_n \rangle$ as an infinite set of reals indexed by the positive integers, i.e. $\{a_1, a_2, \ldots, a_k, \ldots\}$. For example $\langle n^2 \rangle$ is the sequence that starts $\langle 1, 4, 9, \ldots \rangle$. Also $\left\langle 1 - \frac{1}{n} \right\rangle$ starts out $\langle 0, 1/2, 2/3, 3/4, \ldots \rangle$. We want to be able to deal with sequences that get closer and closer to a particular number. That means, the further we go out into the sequence, the closer we should get to that number. The formal definition of the limit of a sequence is this:

Definition. The sequence $\langle a_n \rangle$ approaches a limit l if given any $\varepsilon > 0$ there exists a positive integer N such that for all $n > N$ we have $|a_n - l| < \varepsilon$.

Note that the inequality $|a_n - l| < \varepsilon$ is equivalent to $-\varepsilon < a_n - l < \varepsilon$ which is the same as saying $l - \varepsilon < a_n < l + \varepsilon$.

As an example, we see that the sequence $\left\langle 2 - \frac{1}{n} \right\rangle$ approaches 2 as a limit, because given any $\varepsilon > 0$, we merely choose N so large that $\frac{1}{N} < \varepsilon$. Then for any $n > N$ we have

$\frac{1}{n} < \frac{1}{N} < \varepsilon$ so $-\frac{1}{n} > -\frac{1}{N} > -\varepsilon$. Adding 2 and comparing,

$$2+\varepsilon \ > \ 2 \ > \ 2-\frac{1}{n} \ > \ 2-\varepsilon\,.$$

We find that $2+\varepsilon > 2-\frac{1}{n} > 2-\varepsilon$ or equivalently $\left|\left\{2-\frac{1}{n}\right\}-2\right| < \varepsilon$. This proves the limit is 2.

We use the notation $\lim_{n\to\infty} a_n = l$, or $a_n \to l$ as $n \to \infty$ to express the fact that the limit of the sequence $\langle a_n\rangle$ is l. Another variation of the first expression puts the $n\to\infty$ completely below the text, i.e., $\displaystyle\lim_{n\to\infty} a_n = l$.

Limits have *linear* type properties. That is, if $\lim_{n\to\infty} a_n = l$ and $\lim_{n\to\infty} b_n = m$ then for c a constant

$$\lim_{n\to\infty}(a_n+b_n) \ = \ l+m\,, \qquad \lim_{n\to\infty}(c\,b_n) \ = \ c\,m\,.$$

Limits also behave nicely under products and quotients. Suppose $\langle a_n\rangle$ and $\langle b_n\rangle$ are sequences with limits l and m as above. Then

$$\lim_{n\to\infty}(a_nb_n) \ = \ l\,m\,, \qquad \lim_{n\to\infty}\frac{a_n}{b_n} \ = \ \frac{l}{m} \text{ for } m \neq 0\,.$$

It can be shown that a sequence can have at most one limit. Also it should be noted that the limit of a constant sequence is just that constant. Thus for the sequence that consists of only the number 3, we have $\lim_{n\to\infty} 3 = 3$.

There is also what is called the Sandwich Theorem for sequences. Suppose there are two sequences $\langle a_n\rangle$ and $\langle b_n\rangle$, which both have the limit l. Suppose $\langle x_n\rangle$ is another sequence for which $a_n \le x_n \le b_n$ for all n. Then the Sandwich Theorem states that $\lim_{n\to\infty} x_n$ must also equal l.

If a sequence does not converge, we say that it diverges. A sequence $\langle a_n\rangle$ diverges to plus infinity (or just infinity) if for any G there exists N such that for all $n > N$, $a_n > G$. We write $\lim_{n\to\infty} a_n = \infty$ or $+\infty$ if we wish to emphasize the matter.

A sequence $\langle a_n\rangle$ diverges to minus infinity if for any g there is an N such that for all $n > N$, $a_n < g$. We write $\lim_{n\to\infty} a_n = -\infty$.

For example, $\langle n\rangle$ diverges to plus infinity, as does $\langle n^2\rangle$.

Exercises.

1. Using the definition of a limit, show that $\lim_{n\to\infty} \dfrac{1}{n} = 0$.

Solution. Given $\varepsilon > 0$ we can find an N sufficiently large so that $\dfrac{1}{N} < \varepsilon$. Then for any positive integer n with $n > N$ we have $0 < \dfrac{1}{n} < \dfrac{1}{N} < \varepsilon$. Weakening the left bound gives $-\varepsilon < \dfrac{1}{n} < \varepsilon$ which implies $\left|\dfrac{1}{n} - 0\right| < \varepsilon$ for all $n > N$. That proves $\lim_{n\to\infty} \dfrac{1}{n} = 0$. ■

2. Use the previous exercise and the properties of limits to show that $\lim_{n\to\infty} \dfrac{1}{n^2} = 0$.

Solution. There are a few ways to do this. One way is to note that we can use the product of limits as follows.

$$\lim_{n\to\infty} \frac{1}{n^2} = \left(\lim_{n\to\infty} \frac{1}{n}\right) \cdot \left(\lim_{n\to\infty} \frac{1}{n}\right) = 0 \cdot 0 = 0\,.$$

Another way is to use the Sandwich Theorem. We have $0 \le \dfrac{1}{n^2} \le \dfrac{1}{n}$. From exercise 1, we know $\lim_{n\to\infty} \dfrac{1}{n} = 0$. Also the limit of the constant sequence $< 0 >$ is also 0. Thus by the Sandwich Theorem $\lim_{n\to\infty} \dfrac{1}{n^2} = 0$. ■

3. Prove that $\lim_{n\to\infty} \dfrac{1}{n^k} = 0$ for all integers $k \ge 1$.

4. Find the limit.

$$\lim_{n\to\infty} \frac{2\,n+1}{n-1}\,.$$

Solution. The technique to doing problems of this nature is to find the highest exponent of n occurring in the problem and to divide the numerator and denominator by it. Dividing the numerator and denominator by n^1, and using the results about limits of sums, products and quotients, we have

$$\begin{aligned}\lim_{n\to\infty}\frac{2\,n+1}{n-1} &= \lim_{n\to\infty}\frac{2+\dfrac{1}{n}}{1-\dfrac{1}{n}}\\ &= \frac{\lim_{n\to\infty}\left(2+\dfrac{1}{n}\right)}{\lim_{n\to\infty}\left(1-\dfrac{1}{n}\right)}\\ &= \frac{\lim_{n\to\infty}2\;+\;\lim_{n\to\infty}\dfrac{1}{n}}{\lim_{n\to\infty}1\;-\;\lim_{n\to\infty}\dfrac{1}{n}}\\ &= \frac{2+0}{1-0} \;=\; 2\,.\end{aligned}$$

The limit is 2. ■

5. Find the limit

$$\lim_{n\to\infty}\frac{7\,n^3-3\,n^2+1}{-2\,n^3+15\,n+1}\,.$$

6. Find the limit

$$\lim_{n\to\infty}\frac{3n^5+7n^3+12n+4}{9n^6+5n^2+2}\,.$$

7. Find the limit

$$\lim_{n\to\infty}\frac{4n^3-n^2-2n+6}{8n^2-7n-5}\,.$$

8. Show that the limit of $\langle 1/\sqrt{n}\rangle$ is 0 as $n\to\infty$.

Solution. We have to go back to the definition of a limit in this case. Given $\varepsilon > 0$ we find a large positive number N such that $\dfrac{1}{N} < \varepsilon^2$. Then for all integers $n > N$ we have $\dfrac{1}{n} < \dfrac{1}{N} < \varepsilon^2$, or equivalently $\dfrac{1}{\sqrt{n}} < \dfrac{1}{\sqrt{N}} < \varepsilon$. Therefore $-\varepsilon < \dfrac{1}{\sqrt{n}} < \varepsilon$ which yields $\left|\dfrac{1}{\sqrt{n}} - 0\right| < \varepsilon$. Thus the limit in this case is 0. ■

9. Prove that a sequence $\langle a_n\rangle$ can have at most one limit. (Hint: Suppose it has two limits, l and m. Then $l \neq m$. Choose ε carefully to obtain a contradiction.)