Magooey's Math Problems

Limits of Functions and Continuity

Synopsis. We will make an analogue of the concept of limit, but this time in the case of a function. Suppose $f(x)$ is a function that takes real numbers into real numbers. We want to be able to say that $f(x)$ approaches a specific value m, when x gets closer and closer to a given number a. For example, $f(x) = x^3 - x^2$ should get closer and closer to the value $64 - 16 = 48$ as x gets closer and closer to 4. To put matters on a rigorous basis, mathematicians have accepted the following definition.

Definition. The function $f(x)$ approaches a limit m as x approaches a if for all $\varepsilon > 0$ there exists a $\delta > 0$ for which $|\, f(x) - m\,| < \varepsilon$ for all x satisfying $0 < |\, x - a\,| < \delta$.

In other words, $f(x)$ is within ε of the value m as long as x is sufficiently close to (but not equal to) a. We write this as $\lim_{x\to a} f(x) = m$, or $f(x) \to m$ as $x \to a$. As with sequences, a variation of the notation has the $x \to a$ part underneath the lim portion of the formula, e.g., $\displaystyle\lim_{x\to a} f(x) = m$.

The definition can also be interpreted in this manner: there exists an open interval around a where the value of $f(x)$ does not differ from m by more than ε, except possibly at a itself.

This definition is called the ε–δ definition of limit.

Here, by open interval, we mean the set of all real x such that $b < x < c$ for $b < c$ real numbers. We use the notation $(b,\, c)$ to indicate the open interval just described. For the definition of limit, the open interval around a would be $(a - \delta,\, a + \delta)$.

In addition there are what are called closed intervals. In this case, we consider the set of all real x for which $b \le x \le c$. The notation $[b,\, c]$ is used to denote the closed interval just described.

As an exercise, we will show that the limit of $f(x) = 4\,x + 7$ as x approaches 2 is 15. Given an $\varepsilon > 0$ we wish to find a $\delta > 0$ so that $f(x)$ is within ε of 15 when x lives in the open interval $(2-\delta,\, 2+\delta)$. So $2-\delta < x < 2+\delta$. Then multiplying by 4 and adding 7 throughout this inequality we have

$$\begin{aligned} 4\cdot 2 - 4\delta + 7 \;&<\; 4x+7 \;<\; 4\cdot 2 + 4\cdot\delta + 7 \\ -4\delta \;&<\; 4x+7-15 \;<\; 4\delta \\ |4x+7-15| \;&<\; 4\delta\,. \end{aligned}$$

Choose any δ satisfying $\delta < \varepsilon/4$. Then by the above inequalities, $f(x)$ is within $4\delta < \varepsilon$ of 15 for x in the range $2-\delta < x < 2+\delta$. That proves it.

Limits of functions satisfy properties similar to those of limits of sequences. Suppose $f(x)$ and $g(x)$ are functions and c is a constant. Then if $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$ it follows that

$$\lim_{x\to a} c\cdot f(x) \;=\; c\cdot l\,, \qquad \lim_{x\to a} (f(x)+g(x)) \;=\; l+m\,.$$

We call these the linear type properties for limits. In addition, limits behave nicely under multiplication and division.

$$\lim_{x\to a} f(x)\cdot g(x) \;=\; l\cdot m\,, \qquad \lim_{x\to a} \frac{f(x)}{g(x)} \;=\; \frac{l}{m} \;\text{ provided } m\neq 0\,.$$

It is an exercise to show that every polynomial of the first degree, $f(x) = b\,x + c$, with b, c constant, satisfies

$$\lim_{x\to a} f(x) \;=\; f(a)$$

for each real number a. There is also a Sandwich Theorem for limits of functions.

Theorem. *If $f(x)$ and $g(x)$ are functions which approach the limit m as $x\to a$ and $h(x)$ is a function such that on an open interval around a we have*

$$f(x) \;\le\; h(x) \;\le\; g(x)$$

except possibly at $x=a$, then $\lim_{x\to a} h(x) = m$ also.

We define a function to be continuous at a if $\lim_{x\to a} f(x) = f(a)$. Since all functions of the first degree are continuous at all real a, it follows from the product rule for limits, that all powers of x are also continuous. Just take $g(x) = x^n = x\cdot x\cdot\dots\cdot x$. Knowing that $f(x) = x$ is continuous, using the product property for limits and induction, it follows that $g(x) = x^n$ is also continuous.

But then the linear properties of limit immediately imply that any polynomial in x is continuous. The division property of limits then proves that functions that are the ratio of two polynomials are continuous at all points where the denominator is not zero. So, for example $\dfrac{x^2+1}{x-1}$ is continuous for all values of x except for 1.

Functions that can be written as the ratio of two polynomials are called rational functions.

We give a typical example of a function that is not continuous everywhere. Suppose postage costs 30¢ per ounce or fraction thereof. Then a letter weighing 1.99 ounces will cost 60¢ to mail, while a letter weighing 2.01 ounces will cost 90¢ to mail. The cost of postage jumps at each integer number of ounces. The function that is the cost of postage per ounce is not continuous at positive integer values.

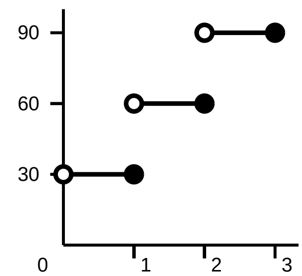

Fig. 1

We can also deal with limits as x goes to $\pm\infty$. We define $\lim_{x\to\infty} f(x) = m$ if for all $\varepsilon > 0$ there T such that for all $x > T$ we have $|\, f(x) - m \,| < \varepsilon$. For example $\lim_{x\to\infty}\left(1+\dfrac{1}{x^2}\right) = 1$.

Similarly we define $\lim_{x\to-\infty} f(x) = m$ if for all $\varepsilon > 0$ there exists t such that for all $x < t$ we have $|\, f(x) - m \,| < \varepsilon$.

Exercises.

1. Show that $f(x) = b\,x + c$ where b and c are constants, satisfies $\lim_{x\to a} f(x) = f(a)$.

Solution. Fix a. Given $\varepsilon > 0$ we wish to find $\delta > 0$ for which $f(x)$ is within ε of $f(a)$ when $0 < |\, x - a \,| < \delta$. We compute

$$|\, f(x) - f(a) \,| \;=\; |\, b\,x + c - (b\,a + c) \,| \;=\; |\, b(x-a) \,| \;=\; b\,|\, x - a \,|\,.$$

There are two cases. If $b = 0$ then $f(x) = c$ a constant, so any δ will do, If $b \neq 0$ then take $\delta = \varepsilon/b$. This gives

$$|\, f(x) - f(a) \,| \;=\; b\,|\, x - a \,| \;<\; b\,\frac{\varepsilon}{b} \;=\; \varepsilon\,.$$

This completes the proof. ■

2. Show that the limit of $f(x) = x^2$ as x approaches 3 is 9 using the definition of limit.

Solution. Given an $\varepsilon > 0$ we wish to find a $\delta > 0$ so that $f(x)$ is within ε of 9 when x lives in the open interval $(3-\delta,\, 3+\delta)$ except possibly at 3. So $3 - \delta < x < 3 + \delta$. Squaring we find

$$\begin{aligned} 9 - 2\,\delta + \delta^2 \;&<\; x^2 \;<\; 9 + 2\,\delta + \delta^2 \\ -2\,\delta - \delta^2 \;<\; -2\,\delta + \delta^2 \;&<\; x^2 - 9 \;<\; 2\,\delta + \delta^2 \\ \left| x^2 - 9 \right| \;&<\; 2\,\delta + \delta^2\,. \end{aligned}$$

Given ε we need only find a δ such that $2\,\delta + \delta^2 < \varepsilon$. We can choose any δ such that $2\,\delta \;<\; \dfrac{\varepsilon}{2}$ and $\delta^2 < \varepsilon/2$. This is accomplished by choosing any δ less that both $\varepsilon/4$ and $\sqrt{\varepsilon/2}$. Then for such a δ

$$\left| x^2 - 9 \right| \;<\; 2\,\delta + \delta^2 \;<\; \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \;=\; \varepsilon\,.$$

This proves the limit is 9. ■

3. Show that $f(x) = \sqrt{x}$ is continuous for all $x > 0$.

4. What value will make $f(x)$ continuous at $x = 2/3$ if

$$f(x) \;=\; \frac{3\,x^3 - 2\,x^2 + 12\,x - 8}{3\,x - 2} \qquad \text{for } x \neq \frac{2}{3}\,.$$

Solution. Let us factor the numerator.

$$3\,x^3 - 2\,x^2 + 12\,x - 8 \;=\; x^2(3\,x - 2) + 4(3\,x - 2) \;=\; (x^2 + 4)(3\,x - 2)\,.$$

Thus $f(x) = x^2 + 4$ for all $x \neq 2/3$. Since polynomials are continuous, the value that will make f continuous at $x = 2/3$ is $(2/3)^2 + 4 = 40/9$. ■

5. Find all values of c that make $f(x)$ continuous at $x = -2$, where

$$f(x) \;=\; \begin{cases} c^2 x + (2c + 5) & x \leq -2 \\ c\,x^2 - 2\,c\,x + 3\,c & x > -2\,. \end{cases}$$

6. Find all values of c that make f a continuous function.

$$f(x) \;=\; \begin{cases} c\,x + \dfrac{2}{c\,x} & x \leq 4 \\ 4\,c\,\sqrt{x} + 1 & x > 4\,. \end{cases}$$

7. Show that $\lim_{x \to \infty} \left(1 + \dfrac{1}{x^2}\right) = 1$

Solution. Given $\varepsilon > 0$ we wish to find T large enough to force $\left|(1 + (1/x^2)) - 1\right| < \varepsilon$ for all $x > T$. This simplifies to having $1/x^2 < \varepsilon$ or equivalently $x > 1/\sqrt{\varepsilon}$. Choose any $T > 1/\sqrt{\varepsilon}$. Then going backwards in the above argument, we have for any $x > T$ that $\left|(1 + (1/x^2)) - 1\right| < \varepsilon$. ■

8. Use the Sandwich Theorem to show that $\lim_{x \to 0} f(x) = 4$ if $f(x) = 4 + x \cdot \sin(x)$.