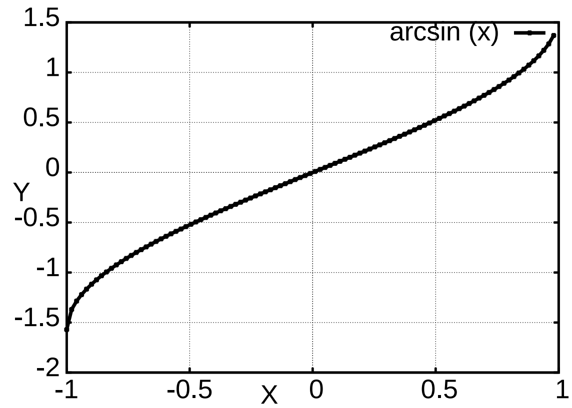
Magooey's Math Problems

Inverses of Trigonometric Functions

Synopsis. As a special case of inverse functions, we can consider the inverse trigonometric functions. Here, the domain and ranges of the functions become very important. Since the trigonometric functions are periodic, many values of x will produce the same values of $\sin(x)$. The same holds for $\cos(x)$, $\tan(x)$, etc. In order to get an inverse, we must restrict each trigonometric function to a specific interval where it takes on each possible value only once. For example, if we restrict x to be in the close interval $[-\pi/2, \pi/2]$ then $\sin(x)$ takes on each value in the interval $[-1, 1]$ exactly once. Hence we can define an inverse to the sine function when restricted to these closed intervals.



We define this function $\sin^{-1}(x)$, or in older notation, $\arcsin(x)$. The -1 is read "inverse" and $\sin^{-1}(x)$ is understood not to be the function $1/\sin(x)$. As $\sin(x)$ maps $[-\pi/2, \pi/2]$ to $[-1, 1]$, it follows that $\arcsin(x)$ acts in the opposite direction. Namely $\arcsin(x)$ maps $[-1, 1]$ to $[-\pi/2, \pi/2]$. Clearly, $\arcsin(x)$ is the angle in the domain $[-\pi/2, \pi/2]$ for which the sine function has the value x. In particular, we note that

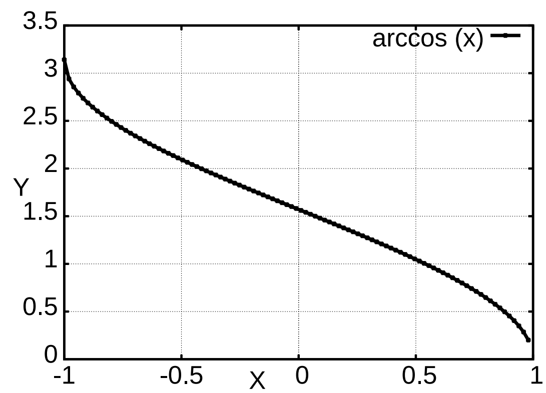
$$\begin{aligned}
\arcsin(0) = 0, \quad \arcsin\left(\frac{1}{2}\right) &= \pi/6, & \arcsin\left(-\frac{1}{2}\right) &= -\pi/6 \\
\arcsin\left(\frac{\sqrt{2}}{2}\right) &= \pi/4, & \arcsin\left(-\frac{\sqrt{2}}{2}\right) &= -\pi/4 \\
\arcsin\left(\frac{\sqrt{3}}{2}\right) &= \pi/3, & \arcsin\left(-\frac{\sqrt{3}}{2}\right) &= -\pi/3 \\
\arcsin(1) &= \pi/2, & \arcsin(-1) &= -\pi/2 \,.
\end{aligned}$$

In order to find the derivative of the function $\arcsin(x)$ we use the Chain Rule. Set $y = \arcsin(x)$. Then $\sin(y) = x$. Note that $\sin(\arcsin(x)) = x$ by the fact of being inverse functions. Applying the Chain rule to this last equation yields

$$\begin{aligned} \cos(\arcsin(x)) \cdot \frac{d}{dx}(\arcsin(x)) &= 1 \\ \cos(y) \cdot \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\cos(y)} . \end{aligned}$$

Since $x = \sin(y)$ and $\sin^2(y) + \cos^2(y) = 1$ we find that $1/\cos(y) = \pm 1/\sqrt{1-x^2}$. Since cosine is always positive in the region $[-\pi/2, \pi/2]$ we must take the positive square root. Our conclusion is

$$\frac{d}{dx}(\arcsin(x)) = \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} .$$

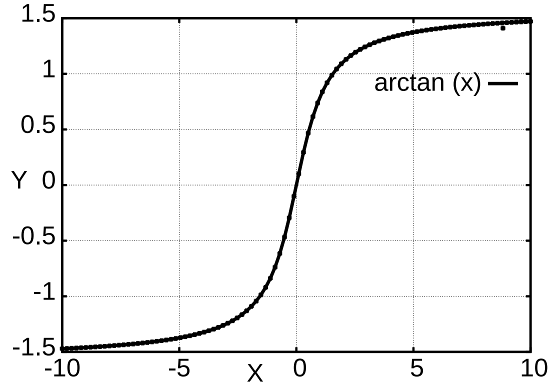


For the inverse of the cosine function, we restrict cosine to the closed interval $[0, \pi]$. Since cosine maps this interval to $[-1, 1]$ we define its inverse function $\arccos(x)$ or $\cos^{-1}(x)$ on the domain $[-1, 1]$ and mapping onto $[0, \pi]$. We follow the same procedure to find the derivative of $\arccos(x)$ as we did for $\arcsin(x)$. Set $y = \arccos(x)$. Then $\cos(y) = x$. Note that $\cos(\arccos(x)) = x$ by the fact of being inverse functions. Again apply the Chain rule to obtain

$$\begin{aligned} -\sin(\arccos(x)) \cdot \frac{d}{dx}(\arccos(x)) &= 1 \\ -\sin(y) \cdot \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= -\frac{1}{\sin(y)} . \end{aligned}$$

In this case, since $x = \cos(y)$ and $\sin^2(y) + \cos^2(y) = 1$ we find that $-1/\sin(y) = \pm 1/\sqrt{1-x^2}$. Since sine is always positive in the region $[0, \pi]$ and arccosine is decreasing (since cosine is decreasing on $[0, \pi]$) we must take the negative square root. Our conclusion is

$$\frac{d}{dx}(\arccos(x)) = \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}} .$$



It is also necessary to consider arctangent, the inverse of the tangent function. In this case we restrict the domain of the tangent function to the open interval $(\pi/2,\, \pi/2)$. The range of the tangent function is $(-\infty,\, \infty)$. Therefore the arctangent maps $(-\infty,\, \infty)$ to the open interval $(\pi/2,\, \pi/2)$. To find the derivative of arctangent, set $y = \arctan(x)$. Then $\tan(y) = x$ and $\tan(\arctan(x)) = x$. Differentiating the last expression, we obtain

$$\begin{aligned} \sec^2(\arctan(x)) \cdot \frac{d}{dx}(\arctan(x)) &= 1 \\ \sec^2(y) \cdot \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\sec^2(y)}\,. \end{aligned}$$

This follows from the fact that the derivative of $\tan(y)$ is $\sec^2(y)$. However, we note that $\sec^2(y) = 1 + \tan^2(y) = 1 + x^2$. Substituting, we find

$$\frac{d}{dx}(\arctan(x)) = \frac{dy}{dx} = \frac{1}{1+x^2}\,.$$

The other three trigonometric functions, cotangent, secant and cosecant, have inverses but these are obscure. There is still debate on the best domains for some of these functions. One has to take care in those rare instances when inverse functions to the above three trigonometric functions crop up.

Exercises.

1. Find the derivative of $h(x) = \arcsin(5\,x)$.

Solution. We apply the Chain Rule to $(g \circ f)(x)$ with $f(x) = 5\,x$ and $g(x) = \arcsin(x)$.

$$\frac{d}{dx}(g \circ f)(x) = g'(f(x)) \cdot f'(x) = g'(5\,x) \cdot 5 = \frac{1}{\sqrt{1-(5\,x)^2}} \cdot 5 = \frac{5}{\sqrt{1-25\,x^2}}\,.$$

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2. Find the derivative of $h(x) = \arccos(3\,x^2)$.

3. Find the derivative of $h(x) = \arcsin(2 - x)$.

Solution. We apply the Chain Rule to $(g \circ f)(x)$ with $f(x) = 2 - x$ and $g(x) = \arcsin(x)$.

$$\begin{aligned} \frac{d}{dx}(g \circ f)(x) &= g'(f(x)) \cdot f'(x) \;=\; g'(2-x) \cdot (-1) \\ &= \frac{1}{\sqrt{1-(2-x)^2}} \cdot (-1) \;=\; -\frac{1}{\sqrt{-3+4\,x-x^2}}\,. \end{aligned}$$

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4. Find the derivative of $h(x) = \arctan(3x)$.

5. Find a function whose derivative is $\dfrac{1}{\sqrt{1-4\,x^2}}$

Solution. This appears to be close in form to the derivative of $\arcsin(x)$. By considering the previous examples. we may reasonably try a function of the form $h(x) = a\,\arcsin(b\,x)$ for a and b constants. The derivative of $h(x)$ can be computed by the Chain Rule.

$$\frac{d}{dx}h(x) \;=\; a\,\frac{1}{\sqrt{1-b^2x^2}} \cdot b \;=\; \frac{a\,b}{\sqrt{1-b^2x^2}}\,.$$

We may choose $b = 2$ and that forces $a = 1/2$. The answer is $\dfrac{1}{2}\,\arcsin(2\,x)$. ■

6. Find a function whose derivative is $\dfrac{1}{1+49\,x^2}$.

7. Find a function whose derivative is $\dfrac{1}{25+4\,x^2}$.

8. Simplify the expression $\cos(2\,\arccos(x))$. The result is a polynomial in x.

Solution. Let $y = 2\,\arccos(x)$. Then we are looking for $\cos(y)$. We have $y/2 = \arccos(x)$, which implies $\cos(y/2) = x$. Now $\cos(y) = \cos(2\,(y/2)) = 2\,\cos^2(y/2) - 1$. Substituting in for x we find $\cos(y) = 2\,x^2 - 1$. ■

9. Simplify the expression $\cos(3\,\arccos(x))$. The result is a polynomial in x.

10. Find a function whose derivative is $\dfrac{1}{\sqrt{-15+8\,x-x^2}}$.