Magooey's Math Problems

Topics in Inverse Functions

Synopsis. Once we have the concepts of composition of functions and the Chain Rule, we can apply these two ideas together in a useful special case, namely inverse functions. As an example, suppose we have the function $f(x)$ defined on the closed interval $[0,\ 1]$ and mapping into the closed interval $[5,\ 7]$ with the formula $f(x) = 2\,x + 5$. We wish to find a function g that undoes the action of f. That is, we wish to find a g such that $(g \circ f)(x) = g\Big(\, f(x)\,\Big) = x$ for all $x \in [0,\ 1]$. The method for accomplishing this is to write $y = f(x) = 2\,x + 5$ and then solve for x in terms of y. We get

$$\begin{aligned} y &= 2\,x + 5 \\ y - 5 &= 2\,x \\ \frac{y-5}{2} &= x \\ x &= g(y) \ = \ \frac{y-5}{2}\,. \end{aligned}$$

We set g to be the function mapping $[5,\ 7]$ to $[0,\ 1]$ by the equation $g(x) = (x-5)/2$. Then

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \ = \ g(2\,x+5) \ = \ \frac{(2\,x+5)-5}{2} \ = \ \frac{2\,x}{2} \ = \ x\,, \\ (f \circ g)(x) &= f(g(x)) \ = \ f\left(\frac{x-5}{2}\right) \ = \ 2\,\frac{x-5}{2} + 5 \ = \ x - 5 + 5 \ = \ x\,. \end{aligned}$$

In the general case, suppose f is a function taking the set X into the set Y, and that there exists a function g that undoes the action of f. That is, g maps Y into X with these properties:

$$\begin{aligned} (g \circ f) : X \mapsto X \ \text{ and } \ (g \circ f)(x) &= x \ \text{ for all } \ x \in X, \\ (f \circ g) : Y \mapsto Y \ \text{ and } \ (f \circ g)(y) &= y \ \text{ for all } \ y \in Y. \end{aligned}$$

Then we say g is the inverse function of f and write $g = f^{-1}$. This is to be understood as distinct from the reciprocal of f which is the function $\dfrac{1}{f(x)}$.

Another example may help clarify matters. Set $f(x) = (x+1)/x$. This is defined on all of the reals except $x = 0$. Then solving $y = (x+1)/x$ for x we get $x = 1/(y-1)$. So the inverse function g is defined for all real $y \neq 1$, and we may write either $g(y) = 1/(y-1)$ or exchanging variables $g(x) = 1/(x-1)$. The point here is that we sometimes have to be careful about the domains and ranges of functions and their inverses.

Verifying the properties of inverse function we should compute $(f \circ g)(x)$ and $(g \circ f)(x)$. We do the first of these, and leave the second as an exercise for the reader.

$$\begin{aligned}(f \circ g)(x) &= f\Big(g(x)\Big) = f\left(\frac{1}{x-1}\right) \\ &= \frac{\dfrac{1}{x-1}+1}{\dfrac{1}{x-1}} = \frac{\dfrac{x}{x-1}}{\dfrac{1}{x-1}} = \frac{x}{1} = x\,.\end{aligned}$$

This is valid for all $x \neq 1$.

Suppose that f maps X into Y and has an inverse function g which takes Y to X Suppose both f and g are differentiable. Then differentiating both sides of the equation $(g \circ f)(x) = x$ and applying the Chain Rule, we find that

$$g'(f(x)) \cdot f'(x) = 1 \quad \text{or} \quad g'(f(x)) = \frac{1}{f'(x)}\,.$$

Writing $y = f(x)$ and $x = g(y)$ we get the intuitive looking version of the above equation by changing into Liebnitz notation.

$$g'(y) = \frac{1}{f'(x)}\,, \qquad \frac{dx}{dy} = \frac{1}{dy/dx}\,, \qquad \frac{dx}{dy} \cdot \frac{dy}{dx} = 1\,.$$

We may verify this equation relating the derivatives of functions inverse to each other in the cases considered above. For $f(x) = 2\,x + 5$ and $g(x) = f^{-1}(x) = (x-5)/2$ we have $f' = 2$ and $g' = 1/2$ so $g'(y) \cdot f'(x) = 1$.

For $f(x) = (x+1)/x$ and $g(x) = f^{-1}(x) = 1/(x-1)$ we have

$$\begin{aligned}f'(x) &= \frac{x \cdot \mathrm{D}(x+1) - (x+1) \cdot \mathrm{D}(x)}{x^2} = \frac{x - (x+1)}{x^2} = \frac{-1}{x^2}\,, \\ g'(x) &= \frac{1 \cdot \mathrm{D}(x-1) - 0}{(x-1)^2} = \frac{-1}{(x-1)^2}\,.\end{aligned}$$

However, we must calculate $g'(y) = g'\Big(f(x)\Big)$ which equals

$$g'\left(\frac{x+1}{x}\right) = \frac{-1}{\left(\dfrac{x+1}{x} - 1\right)^2} = \frac{-1}{\left(\dfrac{1}{x}\right)^2} = -x^2\,.$$

It follows that $g'(y) \cdot f'(x) = 1$.

Exercises.

1. Let $f(x) = 4x - 11$ be a function mapping the reals into the reals. Find $g = f^{-1}$, and verify that g is the inverse function.

Solution. Set $y = 4x - 11$. Then $y + 11 = 4x$ so $x = g(y) = (y + 11)/4$. Exchanging x and y we find $g(x) = (x + 11)/4$.

We check that $(g \circ f)(x) = x$ and also $(f \circ g)(x) = x$.

$$(g \circ f)(x) = g\Big(f(x)\Big) = g(4x - 11) = \frac{(4x - 11) + 11}{4} = \frac{4x}{4} = x,$$

$$(f \circ g)(x) = f\Big(g(x)\Big) = f\left(\frac{x + 11}{4}\right) = 4 \cdot \frac{x + 11}{4} - 11 = x + 11 - 11 = x.$$

■

2. Let $f(x) = 1/(2x)$ be a function mapping the positive real numbers into the positive real numbers. Find the inverse of f, and prove that this function has the required properties.

Solution. Set $y = 1/(2x)$. Then $y \cdot x = 1/2$ so we find $x = g(y) = 1/(2y)$. Thus $g(x) = 1/(2x)$ for all positive real numbers. In other words $f(x) = g(x)$ for all $x > 0$.

Showing that both $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$ for all x, reduces to verifying $(f \circ f)(x) = x$. But

$$(f \circ f)(x) = f\Big(f(x)\Big) = \frac{1}{2f(x)} = \frac{1}{2\,\dfrac{1}{2x}} = \frac{1}{\dfrac{1}{x}} = x.$$

■

3. Let $f(x) = \sqrt{x}$ be a function mapping the nonnegative real numbers into the nonnegative reals. Find $g = f^{-1}(x)$ and prove it has the requisite properties.

4. Let $f(x) = \frac{5}{2}\sqrt{4 - x^2}$, where $0 \le x \le 2$. Find $f'(x)$, f^{-1}, and $\left(f^{-1}\right)'$. Let $x = g(y) = f^{-1}(y)$. Verify that $g'(y) = 1/f'(x)$. What type of curve does $f(x)$ represent?

5. Find the inverse function of $f(x) = 7x^3 - 2$. Verify that for $g = f^{-1}$ we have $f'(y) \cdot g'(x) = 1$.

Solution. Set $y = f(x) = 7x^3 - 2$. Then $(y+2)/7 = x^3$ so $x = \left(\frac{y + 2}{7}\right)^{1/3}$. So $dy/dx = 21x^2$ while $dx/dy = \frac{1}{3} \cdot \frac{1}{7} \cdot \left(\frac{y + 2}{7}\right)^{-2/3}$ which simplifies to $\frac{1}{21}x^{-2}$. Thus $(dy/dx) \cdot (dx/dy) = 21x^2 \cdot \frac{1}{21}x^{-2} = 1$. ■

6. Find the equation of the tangent to the inverse function of $f(x) = 2\,x^3 + 1$ corresponding to $x = 5/2$ on the curve $f(x)$.