Magooey's Math Problems

Induction

Synopsis. Mathematical induction is a technique used to prove results in mathematics. The idea is that there is a beginning point, the base of the induction, called Statement 1. Then there is the inductive principal or inductive step, which is the claim that if Statement n is true, it follows that Statement $n + 1$ is also true.

If both the above propositions can be justified, then it follows that Statement n is true for all integers $n \geq 1$ The logic is that we have showed that Statement 1 is true. Setting $n = 1$, the inductive principal says "since Statement 1 is true, then Statement $1 + 1$ is true". Thus Statement 2 is true. But now that Statement 2 is true, then Statement $2 + 1$ is true. Continuing onwards, we see that Statement n is true for all positive integers n. Some examples will clarify what this method can do for us.

Variations on the technique of mathematical induction include starting with a base case that is not Statement 1, but another Statement. Another variation is to prove some initial cases, and then assume by induction that the Statements are valid for all cases less than n. Then we wish to show the validity of Statement n.

Certain of the problems require the notation for "n factorial". This is the quantity $n! = 1 \times 2 \times \cdot \times n$ where n is a positive integer. We also define $0! = 1$. We say "n factorial" for the expression $n!$. This notation could be familiar from probability.

Exercises.

1. Prove by induction that for n a positive integer

$$1 + 2 + \cdots + n \ = \ \frac{n(n+1)}{2}.$$

Solution. The base case is $n = 1$, and $1 = \dfrac{1(1+1)}{2}$. So the base case is valid. Suppose the proposition is true for n. Thus we know $1 + 2 + \cdots + n \ = \ \dfrac{n(n+1)}{2}$. We wish to prove the case corresponding to $n+1$, namely

$$1+2+\cdots+n+(n+1) \;=\; \frac{(n+1)(n+2)}{2}\,.$$

However,

$$\begin{aligned} 1+2+\cdots+n+(n+1) &= \frac{n(n+1)}{2}+(n+1) \\ &= (n+1)\left\{\frac{n}{2}+1\right\} \\ &= (n+1)\,\frac{n+2}{2} \end{aligned}$$

so the inductive step is valid, and the proposition is true for all positive integer n. ■

2. Prove by induction that for n a positive integer

$$1+3+5+\cdots+(2\,n-1) \;=\; n^2\,.$$

3. Prove by induction that for n a positive integer

$$1^2+2^2+3^2+\cdots+n^2 \;=\; \frac{n(n+1)(2n+1)}{6}$$

Solution. Let us check the base case $n=1$. $1^2=\dfrac{1\cdot(1+1)\cdot(2\cdot 1+1)}{6}$ so the base case is valid. We suppose the proposition is true for the case of n and wish to prove it true for the case of $n+1$. Thus we assume

$$1^2+2^2+3^2+\cdots+n^2 \;=\; \frac{n(n+1)(2n+1)}{6}$$

and wish to prove

$$1^2+2^2+3^2+\cdots+n^2+(n+1)^2 \;=\; \frac{(n+1)(n+1+1)(2(n+1)+1)}{6}\,.$$

The left side equals

$$\begin{aligned} \frac{n(n+1)(2n+1)}{6}+(n+1)^2 &= (n+1)\left\{\frac{n(2n+1)}{6}+n+1\right\} \\ &= (n+1)\,\frac{2n^2+7n+6}{6} \\ &= (n+1)\,\frac{(2n+3)\,(n+2)}{6}\,. \end{aligned}$$

Therefore the statement holds for all positive integer n. ■

4. Prove by induction that for n a positive integer

$$\frac{n\,(n+1)\,(n+2)}{3}$$

is always an integer.

Solution. Consider the base case $n = 1$. Then $n(n+1)(n+2) = 1 \cdot 2 \cdot 3 = 6$ which is divisible by 3. So the base case is true.

Assume the statement holds for the case of n, we wish to show it is valid for the case of $n+1$. That is, we wish to show $\frac{(n+1)(n+2)(n+3)}{3}$ is an integer. Consider

$$\begin{aligned} \frac{(n+1)(n+2)(n+3)}{3} - \frac{n(n+1)(n+2)}{3} &= \frac{(n+1)(n+2)}{3}\{(n+3)-n\} \\ &= (n+1)(n+2)\,. \end{aligned}$$

Therefore, since $\frac{n(n+1)(n+2)}{3}$ is an integer, so it remains an integer when $(n+1)(n+2)$ is added to it. Hence the inductive step is shown, and the statement is true for all positive integers n. ■

5. Prove by induction that for n a positive integer

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2\,n-1)(2\,n+1)} = \frac{n}{2\,n+1}$$

6. Recall the definition of "n factorial", namely $0! = 1$, $1! = 1$ and for positive integers n, $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$. Prove by induction that for positive integers $n \geq 2$ we have $n! \geq 2^{n-1}$.

Solution. Here we start the induction with the base case of $n = 2$. We note that $2! = 1 \cdot 2 = 2 = 2^{2-1}$, hence the base case is valid.

Suppose that for $n > 2$ we know $n! \geq 2^{n-1}$. We wish to show that $(n+1)! \geq 2^{(n+1)-1}$. However $(n+1)! = n! \cdot (n+1) \geq 2^{n-1} \cdot (n+1) \geq 2^{n-1} \cdot 2 = 2^n$. Thus the inductive step holds and the inequality is true for all integer $n \geq 2$. ■

7. Prove by induction that for positive integer n

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}\,.$$

8. The Fibonacci F_n sequence is defined by the recursive equation,

$$F_{n+2} = F_{n+1} + F_n\,, \qquad n \geq 0,$$

with starting values $F_0 = F_1 = 1$. The first few values of the Fibonacci sequence are 1, 1, 2, 3, 5, 8, 13, ... Prove by induction that for all $n \geq 1$

$$F_n = \frac{1}{2}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{1}{2}\left(\frac{1-\sqrt{5}}{2}\right)^n .$$

Solution. In this situation, we must check the base cases $n = 0$ and $n = 1$ as the recursion depends on two terms. For the case of $n = 0$ we observe that

$$\frac{1}{2}\left(\frac{1+\sqrt{5}}{2}\right)^0 + \frac{1}{2}\left(\frac{1-\sqrt{5}}{2}\right)^0 = \frac{1}{2} + \frac{1}{2} = 1 = F_0.$$

When $n = 1$ we find

$$\frac{1}{2}\left(\frac{1+\sqrt{5}}{2}\right)^1 + \frac{1}{2}\left(\frac{1-\sqrt{5}}{2}\right)^1 = \frac{1}{2}\cdot\frac{1+\sqrt{5}}{2} + \frac{1}{2}\cdot\frac{1-\sqrt{5}}{2} = \frac{1}{2} + \frac{1}{2} = 1 = F_1.$$

We can assume by induction that the formula holds for all values less that $n+2$, and we wish to prove the formula in the case of $n+2$. In particular we have

$$F_n = \frac{1}{2}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{1}{2}\left(\frac{1-\sqrt{5}}{2}\right)^n \quad \text{and} \quad F_{n+1} = \frac{1}{2}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + \frac{1}{2}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} .$$

So we have to add

$$\begin{aligned}
\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} &= \left(\frac{1+\sqrt{5}}{2}\right)^n \left\{1 + \frac{1+\sqrt{5}}{2}\right\} \\
&= \left(\frac{1+\sqrt{5}}{2}\right)^n \frac{3+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^n \cdot \frac{6+2\sqrt{5}}{4} \\
&= \left(\frac{1+\sqrt{5}}{2}\right)^n \left(\frac{1+\sqrt{5}}{2}\right)^2 = \left(\frac{1+\sqrt{5}}{2}\right)^{n+2} .
\end{aligned}$$

Taking conjugates we immediately find

$$\left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} = \left(\frac{1-\sqrt{5}}{2}\right)^{n+2} .$$

Substituting, we observe that

$$F_{n+2} = F_{n+1} + F_n = \frac{1}{2}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} + \frac{1}{2}\left(\frac{1-\sqrt{5}}{2}\right)^{n+2} ,$$

which demonstrates the formula by induction. ■

9. With the Fibonacci sequence as above, prove by induction that for $n \geq 2$

$$F_{n+1}\,F_{n-1} - F_n^2 = (-1)^n .$$

10. Prove by induction that for positive integer n,

$$\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} \ \leq\ 2-\frac{1}{n}.$$

Solution. For the case of $n=1$ we observe that $\frac{1}{1!}=1\leq 2-\frac{1}{1}$, and for $n=2$ we have $\frac{1}{1!}+\frac{1}{2!}=1+1/2\leq 2-1/2$. Now suppose $n>2$. Then we assume the proposition for n and wish to prove the case of $n+1$, in particular

$$\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}+\frac{1}{(n+1)!} \ \leq\ 2-\frac{1}{n+1}.$$

Working on the left side, we find it is less that or equal to $2-\frac{1}{n}+\frac{1}{(n+1)!}$. We need only show that

$$2-\frac{1}{n}+\frac{1}{(n+1)!} \ \leq\ 2-\frac{1}{n+1}.$$

This is equivalent to

$$\frac{1}{(n+1)!} \ \leq\ \frac{1}{n}-\frac{1}{n+1}$$

or

$$\frac{1}{(n+1)!} \ \leq\ \frac{1}{n(n+1)}$$

which clearly holds as long as $n+1\geq 2$ or $n\geq 1$. Therefore, the proposition holds for all positive integer n by induction. ■