Magooey's Math Problems

Implicit Differentiation

Synopsis. Sometimes we have an equation relating x and y in which it is difficult if not impossible to solve for y in terms of x. We can still find $\frac{dy}{dx} = y'(x) = y'$ by using the Chain Rule. We leave the y' term in the resulting equation, and solve for it in terms of x and y. This process is called implicit differentiation. An example will clarify the procedure.

Suppose we wish to find the slope of the tangent to the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ at the point $\left(2\sqrt{5}/3,\ 2\right)$. We could either solve explicitly for y in terms of x and differentiate the resulting expression, or we could use the Chain Rule in the following manner. Differentiation of the entire equation with respect to x yields

$$\frac{2x}{4} + \frac{2yy'}{9} = 0\,.$$

Now solve for y' in terms of the other variables.

$$\begin{aligned} \frac{yy'}{9} &= -\frac{x}{4} \\ y' &= -\frac{9x}{4y} \end{aligned}$$

With the values $x = 2\sqrt{5}/3$ and $y = 2$, the slope of the tangent to the ellipse is

$$y' = -\frac{9 \cdot 2\sqrt{5}/3}{4 \cdot 2} = -\frac{3\sqrt{5}}{4}\,.$$

Exercises.

1. Find the slope of the tangent to the curve $x^2 + xy + 3y^2 = 11$ at the point $(1,\ -2)$.

Solution. We differentiate the given equation to find

$$2x + xy' + y \cdot 1 + 6yy' = 0\,.$$

Solving for y' we observe that $y' = -\frac{2x+y}{x+6y}$ which evaluates in this case to $y' = 0$. ■

2. Find the slope of the tangent to the curve $x^5 + x^3 y^2 + 2xy + y = 11$ at the point $(1,\ 2)$.

3. Find the derivative of the function defined by $x\ \sin(y) - \cos(y) = 1$ at the point $(\sqrt{3},\ \pi/3)$.

Solution. We differentiate to find

$$\begin{aligned} x \cdot \frac{d}{dx}(\sin(y)) + \sin(y) \cdot \frac{d}{dx}x - \frac{d}{dx}(\cos(y)) &= 0 \\ x \cdot \cos(y) \cdot y' + \sin(y) - (-\sin(y) \cdot y') &= 0\,. \end{aligned}$$

Solving for y' we find $y' = -\dfrac{\sin(y)}{x\ \cos(y) + \sin(y)}$. This evaluates to $y' = -1/2$ at the point $(\sqrt{3},\ \pi/3)$. ■

4. Find the equation of the line tangent to the curve $x^3 + 3xy^2 - y^3 = -1$ at the point $(2,\ -3)$.

5. Use implicit differentiation to find the second derivative with respect to x of the curve defined by the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ at the point $(2\sqrt{5}/3,\ 2)$.

6. Find the slope of the curve $\sin(x) \cdot \sin(y) = \sqrt{2}/4$ at the point $(\pi/4,\ \pi/6)$.