Magooey's Math Problems

Higher Derivatives

Synopsis. The second derivative of a function $f(x)$ is the derivative of the derivative $f'(x)$. We use the notation f'', $D^2 f$, or $\frac{d^2 f}{dx^2}$ for the second derivative. The third derivative, fourth derivative, etc., are defined similarly.

The second derivative can also be found using limits, though this quickly becomes a cumbersome process. One limit formula for the second derivative is

$$f''(x) \;=\; \lim_{h\to 0} \frac{f(x+2\,h) - 2\,f(x+h) + f(x)}{h^2}\,.$$

Higher order derivatives are useful in graphing functions. The second derivative can tell when a point on the graph of a function is a local maximum, local minimum or neither. In the last case we call the point an inflection point of the graph. The rules are as follows.

Suppose x is a stationary point for the function $f(x)$. If $f''(x) > 0$ then f has a local minimum at x. If $f''(x) < 0$ then f has a local maximum at x.

For any x, if $f''(x) = 0$ we call x a point of inflection or an inflection point of f.

Higher order derivatives come up in the modeling of various phenomenon, as do first order derivatives.

Exercises.

1. Use the definition of the second derivative to find $f''(x)$ for $f(x) = x^2$ and $g(x) = x^3$.

Solution. Let $f(x) = x^2$, Then

$$\begin{aligned} f(x+2h) - 2\,f(x+h) + f(x) &= (x+2h)^2 - 2\,(x+h)^2 + x^2 \\ &= x^2 + 4\,xh + 4\,h^2 - 2(x^2 + 2\,xh + h^2) + x^2 \\ &= 2\,h^2, \end{aligned}$$

so the limit in the definition of the second derivative becomes $\lim_{h\to 0} \frac{2\,h^2}{h^2}$, or 2.

Now suppose $g(x) = x^3$. Then

$$\begin{aligned} g(x+2h) - 2\,g(x+h) + g(x) &= (x+2h)^3 - 2\,(x+h)^3 + x^3 \\ &= (x^3 + 6\,x^2h + 12\,xh^2 + 8\,h^3) \\ &\qquad - 2\,(x^3 + 3\,x^2h + 3\,xh^2 + h^3) + x^3 \\ &= 6\,xh^2 + 8\,h^3\,. \end{aligned}$$

The limit in the definition of the second derivative becomes $\lim_{h\to 0} \dfrac{6\,xh^2 + 8\,h^3}{h^2}$ which is just $6\,x$. ■

2. Use the definition of the second derivative to find $f''(x)$ for $f(x) = \sqrt{x}$. (Hint: This is not easy. You have to multiply by conjugates twice)

3. Find the second derivative of $f(x) = x^3 + 2\,x$.

Solution. Here we jut apply the standard formulas. $f'(x) = 3\,x^2 + 2$. Thus $f'' = 6\,x$. ■

4. Find the second derivative of $f(x) = x^2 + \dfrac{1}{x^2}$.

5. Find the second derivative of $f(x) = 6\,x^{1/3}$.

6. Let $f(x) = a\,x^2 + b\,x + (a + b + 2)$, where a and b are constants.. Find a and b for which

$$f(x) = \left(\frac{1}{4}x - 1\right) f'(x) + \left(\frac{1}{4}x^2 + 1\right) f''(x)$$

holds identically for all real x.

7. Find the stationary points of $f(x) = x^3 - 6\,x + 3$ and use the second derivative to determine if they are local maxima or local minima.

Solution. We compute $f'(x)$ and $f''(x)$. $f'(x) = 3\,x^2 - 6$ and $f'' = 6\,x$. The stationary points satisfy $3\,x^2 = 6$ or $x^2 = 2$, so they are $x = \pm\sqrt{2}$. We need only compute $f''(\sqrt{2}) = 6\sqrt{2} > 0$ and $f''(-\sqrt{2}) = -6\,\sqrt{2} < 0$. Hence f has a local minimum at $x = \sqrt{2}$ and a local maximum at $x = -\sqrt{2}$. ■

8. Find the stationary points of $f(x) = x^4 - 5\,x^3 + 7\,x^2$ and use the second derivative to determine if they are local maxima or local minima. Also find any points of inflection.

9. Find the local maxima and local minima for $f(x) = \sin(x) - x/2$ on the interval $[-\pi,\, \pi]$.