Magooey's Math Problems

Review of Formulas

Synopsis. We review here formulas from precalculus and other courses that are often utilized in a typical calculus class. These formulas can be classified into a few categories. There are algebraic formulas, logarithmic formulas, geometric formulas and trigonometric formulas.

Some algebraic formulas that should be familiar are

$$(x+y)^2 \;=\; x^2+2\,x\,y+y^2\,, \qquad (x-y)^2 \;=\; x^2-2\,x\,y+y^2\,, \qquad x^2-y^2 \;=\; (x+y)\,(x-y)\,.$$

There are analogous formulas for cubes.

$$(x+y)^3 \;=\; x^3+3\,x^2\,y+3\,x\,y^2+y^3\,, \qquad x^3-y^3 \;=\; (x-y)\,(x^2+x\,y+y^2)\,.$$

Indeed there are formulas for higher powers too. Let the exponent n be a positive integer. Then

$$x^n-y^n \;=\; (x-y)\,(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+xy^{n-2}+y^{n-1})\,.$$

The formula for $(x+y)^n$ is called the binomial formula. To set this up, we must first recall the definition of $n!$, called "n factorial". This is simply the product $1\cdot 2\cdots(n-1)\cdot n$ for n a positive integer. We define $0!=1$. So, for example, $2!=2$, $3!=2\cdot 3=6$ and $5!=2\cdot 3\cdot 4\cdot 5=120$. The factorial number grow very fast as n increases.

Next we need the formula for $\binom{n}{k}$. This is

$$\binom{n}{k} \;=\; \frac{n!}{k!\,(n-k)!}\,.$$

This number happens to have a probability interpretation as the number of combinations of n distinct items taken k at a time. For example, $\binom{4}{1}=4$, $\binom{5}{2}=10$ and $\binom{8}{4}=70$. Then the binomial formula follows.

$$\begin{aligned}(x+y)^n \;&=\; x^n+\binom{n}{1}x^{n-1}y+\binom{n}{2}x^{n-2}y^2+\cdots+\binom{n}{n-1}x\,y^{n-1}+y^n \\ &=\; \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k\,.\end{aligned}$$

In addition, there is the Triangle Inequality, which states that $|x+y| \leq |x|+|y|$ for any real x and y.

The basic logarithmic formula is that if $a^b = c$, we say that $\log_a c = b$, or "log base a of c is b". Here we restrict to a and c being positive real numbers. For example $\log_{10} 1000 = 3$ since $10^3 = 1000$. Also $\log_2 1/4 = -2$ since $2^{-2} = 1/4$.

A useful formula for logarithms is $\log_a c + \log_a d = \log_a(c \cdot d)$. Also $\log_a c - \log_a d = \log_a(c/d)$.

In geometry we deal with triangles, circles, lines, and other shapes and figures. Some of the corresponding formulas are used in various places in calculus. For a triangle, $\triangle ABC$, with corresponding sides of length a, b, c, and area $\mathcal{A}$ we have

$$\mathcal{A} = \frac{1}{2} b\, c \sin(A) = \frac{1}{2} a\, c \sin(B) = \frac{1}{2} a\, b \sin(C).$$

Also there is the Law of Cosines,

$$a^2 = b^2 + c^2 - 2\, b\, c \cos(A),$$

with similar formulas for b^2 and c^2. Let $s = \dfrac{a+b+c}{2}$, the semiperimeter. Then another formula for the area is

$$\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)}\,.$$

The area of a circle of radius r is $\pi\, r^2$ while its circumference is $2\,\pi\, r$. The surface area of a three dimensional sphere of radius r is $4\pi\, r^2$ while its volume is $\dfrac{4}{3}\pi\, r^3$.

There is a Theorem of Ptolemy that states that given a quadrilateral inscribed in a circle with sides, in order, of length a, b, c, d and diagonals of length e, f then $a\,c + b\,d = e\,f$.

We also have formulas for geometrical figures when considered in Cartesian coordinates. A circle centered at the origin of radius r is the set of all points (x, y) which satisfy $x^2 + y^2 = r^2$. An ellipse centered at the origin satisfies an equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,.$$

Taking a and b positive, we call them the semiaxes of the ellipse. The area of an ellipse is $\pi\, a\, b$.

An hyperbola centered at the origin satisfies an equation of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\,, \quad \text{or} \quad \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1\,.$$

If the correct type of hyperbola is rotated 45 degrees, we get the formula $x\,y = c$ where c is a constant.

We can also deal with right circular cones, and right circular cylinders. With radius r and height h, a right circular cylinder has volume $\pi\,r^2 h$, while a right circular cone has volume $\frac{1}{3}\pi\,r^2 h$.

The trigonometric formulas come in handy too. Here are a few.

$$\sin^2(x) + \cos^2(x) = 1\,, \qquad \tan(x) \;=\; \frac{\sin(x)}{\cos(x)}\,,$$
$$\sin\left(\frac{\pi}{2} - x\right) \;=\; \cos(x)\,, \qquad \cos\left(\frac{\pi}{2} - x\right) \;=\; \sin(x)\,.$$

It should be noted that $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$. In other words, sine is an odd function, while cosine is an even function. Here are the addition formulas.

$$\sin(x+y) \;=\; \sin(x)\,\cos(y) + \cos(x)\,\sin(y)\,,$$
$$\cos(x+y) \;=\; \cos(x)\,\cos(y) - \sin(x)\,\sin(y)\,, \quad \tan(x+y) \;=\; \frac{\tan(x)+\tan(y)}{1-\tan(x)\,\tan(y)}\,.$$

We have the double angle formulas

$$\sin(2\,x) \;=\; 2\,\sin(x)\,\cos(x)\,, \quad \cos(2\,x) \;=\; \cos^2(x) \;-\; \sin^2(x)\,, \quad \tan(2\,x) \;=\; \frac{2\,\tan(x)}{1\;-\;\tan^2(x)}\,.$$

There are the other three trigonometric functions, namely cotangent, secant and cosecant. They are defined by

$$\cot(x) \;=\; \frac{1}{\tan(x)}\,, \quad \sec(x) \;=\; \frac{1}{\cos(x)}\,. \quad \csc(x) \;=\; \frac{1}{\sin(x)}\,.$$

There are also half angle formulas. Indeed, any number of formulas are derivable from these.

It is also worthwhile noting some special values for sin, cos, and tan. We know

$\sin(0) = 0$	$\cos(0) = 1$	$\tan(0) = 0$
$\sin(30^o) = 1/2$	$\cos(30^o) = \frac{\sqrt{3}}{2}$	$\tan(30^o) = \frac{\sqrt{3}}{3}$
$\sin(45^o) = \frac{\sqrt{2}}{2}$	$\cos(45^o) = \frac{\sqrt{2}}{2}$	$\tan(45^o) = 1$
$\sin(60^o) = \frac{\sqrt{3}}{2}$	$\cos(60^o) = 1/2$	$\tan(60^o) = \sqrt{3}$
$\sin(90^o) = 0$	$\cos(90^o) = 1$	$\tan(90^o) = +\infty$

Also, we must consider radian measure for angles, as this is typical of calculus. A circle corresponds to an angle of 360^o or $2\,\pi$ radians. Hence 1^o equals $\pi/180$ radians. Since radian measure is assumed in calculus, we would say that $\sin(\pi/4) = \sqrt{2}/2$. This is the same as $\sin(45^o) = \sqrt{2}/2$ if we used degree notation.

Exercises.

1. Prove the double angle formula for $\sin(2\,x)$ from the addition formula for sine.

Solution. Start with the addition formula for sine; $\sin(x+y) = \sin(x)\,\cos(y) + \cos(x)\,\sin(y)$. Then set $x = y$. It follows that $\sin(2\,x) = 2\,\sin(x)\,\cos(x)$. ■

2. Prove the double angle formula for $\cos(2\,x)$ from the addition formula for cosine.

3. Prove the double angle formula for $\tan(2\,x)$ from the addition formula for tangent.

4. Show that $1 + \tan^2(x) = \sec^2(x)$.

Solution. Start with the formula $\sin^2(x) + \cos^2(x) = 1$. Divide both sides of the equation by $\cos^2(x)$. We get

$$\frac{\sin^2(x) + \cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

which implies $\tan^2(x) + 1 = \sec^2(x)$. ■

5. Show that $\cot^2(x) + 1 = \csc^2(x)$.

6. Derive the addition formula for cosine from the addition formula for sine.

Solution. We start with $\sin(x+y) = \sin(x)\,\cos(y) + \cos(x)\,\sin(y)$. Note that

$$\sin\left(\frac{\pi}{2} - x - y\right) = \cos(x+y)\,.$$

So we can apply the addition formula for sine with the arguments $\frac{\pi}{2} - x$ and $-y$. The left side of the addition formula becomes $\cos(x+y)$. The right side of the addition formula becomes

$$\sin\left(\frac{\pi}{2} - x\right)\,\cos(-y) + \cos\left(\frac{\pi}{2} - x\right)\,\sin(-y) \;=\; \cos(x)\,\cos(y) - \sin(x)\,\sin(y)\,.$$

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7. Using the value of $\cos\left(\frac{\pi}{4}\right)$ to find the value of cosine of $\frac{\pi}{8}$.

8. Expand $(2\,x+3)^3$ using the binomial formula.

Solution.

$$\begin{aligned}(2\,x+3)^3 &= (2x)^3 + 3\,(2x)^2(3) + 3\,(2x)(3)^2 + 3^3 \\ &= 8\,x^3 + 36\,x^2 + 54\,x + 27\,.\end{aligned}$$

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9. What is the area of the ellipse with equation $4\,x^2 + 9\,y^2 = 25$.

Solution. Rearrange the formula.

$$\frac{x^2}{25/4} + \frac{y^2}{25/9} \;=\; 1.$$

Therefore the semiaxes are $a = 5/2$ and $b = 5/3$. The area is $\pi\,ab = \dfrac{25}{6}\pi$. ■

10. Solve for x.

$$\log_4 \frac{1}{16} \;=\; 2\,x + 1\,.$$

Solution. $\log_4 \frac{1}{16} = \log_4 4^{-2}$. So we find $-2 = 2\,x + 1$ which gives $x = -3/2$. ■