Magooey's Math Problems

Derivatives of Trigonometric Functions

Synopsis. With the limits of trigonometric function computed previously, we can show that these functions are differentiable in addition to being continuous as long as their values do not increase or decrease beyond all bounds. Firstly let us deal with the derivative of the sine function. We wish to find

$$\lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h}.$$

Using the addition formula for sine we can consider the equivalent expression

$$\begin{aligned}\lim_{h\to 0} \frac{\sin(x)\cos(h)-\cos(x)\sin(h)-\sin(x)}{h} &= \lim_{h\to 0} \frac{\sin(x)(\cos(h)-1)-\cos(x)\sin(h)}{h} \\ &= \lim_{h\to 0}\left\{\sin(x)\,\frac{\cos(h)-1}{h}\right\} + \lim_{h\to 0}\left\{\cos(x)\,\frac{\sin(h)}{h}\right\} \\ &= \sin(x)\lim_{h\to 0}\left\{\frac{\cos(h)-1}{h}\right\} + \cos(x)\lim_{h\to 0}\left\{\frac{\sin(h)}{h}\right\}.\end{aligned}$$

From the section on Limits of Trigonometric Functions, we know that

$$\lim_{h\to 0}\frac{\sin(h)}{h} = 1, \qquad \lim_{h\to 0}\frac{1-\cos(h)}{h} = 0.$$

Substituting into the computation for the derivative of the sine of x, we find that

$$\begin{aligned}\mathrm{D}(\sin(x)) &= \frac{d}{dx}\sin(x) = \sin(x)\lim_{h\to 0}\left\{\frac{\cos(h)-1}{h}\right\} + \cos(x)\lim_{h\to 0}\left\{\frac{\sin(h)}{h}\right\} \\ &= \sin(x)\cdot 0 + \cos(x)\cdot 1 = \cos(x).\end{aligned}$$

We can now find the derivative of $\cos(x)$ by the Chain Rule. Since $\cos(x) = \sin\left(\frac{\pi}{2}-x\right)$ we set $f(x) = \sin(x)$ and $g(x) = \frac{\pi}{2}-x$. We just showed that $f'(x) = \sin'(x) = \cos(x)$, while $g'(x) = -1$. By the Chain Rule

$$\begin{aligned}\mathrm{D}(\cos(x)) &= \frac{d}{dx}\cos(x) = \frac{d}{dx}\sin\left(\frac{\pi}{2}-x\right) = \frac{d}{dx}f\Big(g(x)\Big) \\ &= f'(g(x))\cdot g'(x) = \cos\left(\frac{\pi}{2}-x\right)\cdot(-1) \\ &= -\sin(x).\end{aligned}$$

Note that if we differentiate sine twice, we get back to $-\sin(x)$. Similarly if we differentiate cosine twice we result in $-\cos(x)$.

We can get the derivatives of tangent, cotangent, secant and cosecant by using the quotient rule. For tangent

$$\begin{aligned} \mathrm{D}\ (\tan(x)) &= \mathrm{D}\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{\cos(x)\sin'(x) - \sin(x)\cos'(x)}{\cos^2(x)} \\ &= \frac{\cos^2(x) - \sin(x)\cdot(-\sin(x))}{\cos^2(x)} \\ &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)\,. \end{aligned}$$

For the other cases, we note that

$$\frac{d}{dx}\cot(x) = -\sec^2(x)\,, \quad \frac{d}{dx}\sec(x) = \sec(x)\ \tan(x)\,, \quad \frac{d}{dx}\csc(x) = -\cot(x)\ \csc(x)\,.$$

Exercises.

1. Verify that the derivative of $\sin^2(x) + \cos^2(x)$ is 0. This derivative must be 0 since we know the sum is a constant.

Solution. Let $f(x) = \sin^2(x) + \cos^2(x)$. Then

$$f'(x) = 2\sin(x)\frac{d}{dx}\sin(x) + 2\cos(x)\frac{d}{dx}\cos(x) = 2\sin(x)\cos(x) + 2\cos(x)(-\sin(x)) = 0\,.$$

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2. Find the derivative of $\sin(12\,x)$.

Solution. We use the Chain Rule with $f(x) = \sin(x)$ and $g(x) = 12\,x$. Then $f'(x) = \cos(x)$ and $g'(x) = 12$. Then

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x) = \cos(12\,x) \cdot 12 = 12\ \cos(12\,x)\,.$$

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3. Find the derivative of $\sin(7\,x) + \cos(3\,x)$.

4. Find the derivative of $f(x) = \dfrac{\sin(4\,x)}{\cos(3\,x)}$.

5. Find the derivative of $\tan(2\,x)$.

Solution. We use the Chain Rule with $f(x) = \tan(x)$ and $g(x) = 2\,x$. Then $f'(x) = \sec^2(x)$ while $g'(x) = 2$. So

$$(f \circ g)'(x) \;=\; f'(g(x)) \cdot g'(x) \;=\; \sec^2(2\,x) \cdot 2 \;=\; 2\,\sec^2(2\,x)\,.$$

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6. Find the derivative of $\tan(3\,x) + \sec(3\,x)$.

7. Find the derivative of $\sin(2\,x) \cdot \cos(x)$.

8. Find the derivative of $\dfrac{\sin^2(x)}{\cos^2(x)}$.

9. Verify the formula for the derivative of $\cot(x)$.

Solution.

$$\begin{aligned} \mathrm{D}(\cot(x)) \;&=\; \mathrm{D}\left(\frac{\cos(x)}{\sin(x)}\right) \;=\; \frac{\sin(x)\,\dfrac{d}{dx}(\cos(x)) - \cos(x)\,\dfrac{d}{dx}(\sin(x))}{\sin^2(x)} \\ &=\; \frac{\sin(x)(-\sin(x)) - \cos(x)\cos(x)}{\sin^2(x)} \\ &=\; \frac{-1}{\sin^2(x)} \;=\; -\csc^2(x)\,. \end{aligned}$$

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10. Verify the formulas for the derivatives of $\sec(x)$ and $\csc(x)$.