Magooey's Math Problems

Definition and Properties of the Derivative

Synopsis. We recall the definition of the derivative of a function f at a point x,

$$f'(x) \;=\; \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}\,.$$

Other notations for $f'(x)$ are $D\,f(x)$ and $\dfrac{d\,f}{d\,x}$.

As far as properties of the derivative are concerned, we first note that the derivative obeys *linear* type rules. That is, the derivative of the sum of two differentiable functions is just the sum of the derivatives. The derivative of the product of a constant times a differentiable function is that constant times the derivative. In notation,

$$\begin{aligned} (f(x) + g(x))' &= f'(x) + g'(x) \\ (c \cdot f(x))' &= c \cdot f'(x)\,. \end{aligned}$$

Also, the product rule and quotient rule are handy techniques for computing the derivatives of more complicated functions. Recall that we define the product function $(f\,g)\,(x)$ as the function $f(x) \cdot g(x)$. The product rule says that if $f(x)$ and $g(x)$ are differentiable functions at a point x, then the derivative of $f(x) \cdot g(x)$ is

$$\mathrm{D}\,(f\,g)\,(x) \;=\; \Big(f(x)\,g(x)\Big)' \;=\; f(x) \cdot g'(x) + f'(x) \cdot g(x)\,.$$

The quotient rule says that if $f(x)$ and $g(x)$ are differentiable at a point x and $g(x) \neq 0$ then

$$\left(\frac{f(x)}{g(x)}\right)' \;=\; \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}\,.$$

We could use the notation for the quotient function, $\dfrac{f}{g}\,(x)$ so the left side of the above equation could be written $\mathrm{D}\left(\dfrac{f}{g}\right)\,(x)$.

In particular, we can use the product rule and mathematical induction to show that the derivative of $f(x) = x^n$ is $f'(x) = n\,x^{n-1}$ for n an integer greater than or equal to one. From the linear type properties of the derivative, we can immediately calculate the derivative of a polynomial. For instance the derivative of $5\,x^4 - 7\,x^3 + 2\,x + 13$ is just $4 \cdot 5\,x^3 - 3 \cdot 7\,x^2 + 2 = 20\,x^3 - 21\,x^2 + 2$.

Rational functions can also be differentiated by means of the quotient rule. As an example

$$\frac{d}{dx}\left(\frac{x+1}{x-1}\right) = \frac{(x-1)\cdot 1 - (x+1)\cdot 1}{(x-1)^2} = \frac{-2}{(x-1)^2}\,.$$

A special case would be finding the derivative of $1/x^n$. Using the quotient rule

$$\mathrm{D}\left(\frac{1}{x^n}\right) = \frac{x^n \cdot 0 - 1 \cdot n\,x^{n-1}}{(x^n)^2} = \frac{-n\,x^{n-1}}{x^{2n}} = \frac{-n}{x^{n+1}} = -n\,x^{-n-1}\,.$$

The derivative of a function $f(x)$ at a point x can be interpreted geometrically as the slope of the tangent to the curve f at the point x.

It should be mentioned that a function that is differentiable at a point x is also continuous at that point. However, pathological cases can be constructed where a function is continuous on an interval but has no derivative anywhere on that interval.

Exercises.

1. Use the definition of the derivative to find $f'(x)$ for $f(x) = \sqrt{3x}$.

Solution. We consider the expression $\dfrac{f(x+h) - f(x)}{h} = \dfrac{\sqrt{3(x+h)} - \sqrt{3x}}{h}$ and simplify by multiplying above and below by the conjugate of the numerator.

$$\begin{aligned}\frac{\sqrt{3(x+h)} - \sqrt{3x}}{h} \cdot \frac{\sqrt{3(x+h)} + \sqrt{3x}}{\sqrt{3(x+h)} + \sqrt{3x}} &= \frac{3(x+h) - 3x}{h\,(\sqrt{3(x+h)} + \sqrt{3x})} \\ &= \frac{3h}{h\,(\sqrt{3(x+h)} + \sqrt{3x})} \\ &= \frac{3}{\sqrt{3(x+h)} + \sqrt{3x}}\,.\end{aligned}$$

Now all we have to do is take the limit of the last expression as $h \to 0$. This gives us $\dfrac{3}{2\sqrt{3x}}$ or equivalently $\dfrac{\sqrt{3}}{2\sqrt{x}}$ as the derivative of the function $f(x) = \sqrt{3x}$. ■

2. Use the definition of the derivative to find $f'(x)$ for $f(x) = \dfrac{1}{\sqrt{x}}$.

3. Use the definition of the derivative to find $f'(x)$ for $f(x) = x^3 + 2\,x$.

4. Use the definition of the derivative to find $f'(x)$ for $f(x) = x + \dfrac{1}{x}$.

Solution. We consider the expression $\dfrac{f(x+h)-f(x)}{h} = \dfrac{x+h+\dfrac{1}{x+h}-x-\dfrac{1}{x}}{h}$. Let's simplify the numerator.

$$\begin{aligned} x+h+\frac{1}{x+h}-x-\frac{1}{x} &= h+\frac{1}{x+h}-\frac{1}{x} \\ &= h+\frac{x-x-h}{x(x+h)} \\ &= h-\frac{h}{x(x+h)} \end{aligned}$$

Let us divide by h and take the limit as $h \to 0$.

$$\lim_{h\to 0} \frac{h-\dfrac{h}{x(x+h)}}{h} = \lim_{h\to 0} 1-\frac{1}{x(x+h)} = 1-\frac{1}{x^2}\,.$$

So the derivative is $1 - x^{-2}$. ■

5. Use the definition of the derivative to find $f'(x)$ for $f(x) = x^{1/3}$. (Hint: Recall the identity $A^3 - B^3 = (A-B)\,(A^2 + A\,B + B^2)$)

6. Find the derivative of $x^2\,(x^2+5x+1)$ firstly by using the product rule, and also by multiplying out the product and then differentiating.

Solution. For the product rule, we set $f(x) = x^2$ and $g(x) = x^2 + 5\,x + 1$. Then $f'(x) = 2\,x$ while $g'(x) = 2\,x + 5$. So

$$\begin{aligned} \mathrm{D}(f\,g)(x) &= f(x)\cdot g'(x) + f'(x)\cdot g(x) \\ &= x^2\cdot(2\,x+5) + (2\,x)\cdot(x^2+5\,x+1) \\ &= 2\,x^3+5\,x^2+2\,x^3+10\,x^2+2\,x \\ &= 4\,x^3+15\,x^2+2\,x\,. \end{aligned}$$

Multiplying out we get $f(x)\,g(x) = x^4 + 5\,x^3 + x^2$. We immediately get $\mathrm{D}(f\,g)(x) = 4\,x^3 + 15\,x^2 + 2\,x$. ■

7. Find the derivative of $(x^3+2x^2+1)\cdot(x^2-x)$ firstly using the product rule, and also by multiplying out the product and then differentiating.

8. Find the derivative of $\dfrac{x}{x-1}$.

Solution. Set $f(x) = x$ and $g(x) = x - 1$. Then $f'(x) = 1 = g'(x)$. We can use the quotient rule

$$\begin{aligned} \mathrm{D}\left(\frac{f}{g}\right)(x) &= \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)} \\ &= \frac{(x-1)\cdot 1 - x \cdot 1}{(x-1)^2} \\ &= \frac{-1}{(x-1)^2}\,. \end{aligned}$$

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9. Find the derivative of $\dfrac{x^2+1}{x+1}$.

10. Find the derivative of $\dfrac{x^2+x-1}{x^2-1}$.

11. Use the product rule and induction to show that the derivative of $f(x) = x^n$ is $n\,x^{n-1}$. for n an integer, $n \geq 1$.

Solution. It is easy to see from the definition of the derivative, that $\mathrm{D}(x^1) = 1 = 1\,x^0$. Suppose the statement is true for n. Then $\mathrm{D}(x^n) = n\,x^{n-1}$. It follows, using the product rule, that

$$\begin{aligned} \mathrm{D}(x^{n+1}) &= \mathrm{D}(x^n \cdot x) = x^n \cdot \mathrm{D}(x) + \mathrm{D}(x^n)\cdot x \\ &= x^n \cdot 1 + n\,x^{n-1}\cdot x = (n+1)\,x^n\,. \end{aligned}$$

Thus the statement is true for all $n \geq 1$ by induction. ■