Magooey's Math Problems

The Chain Rule

Synopsis. Recall that the composition of two functions, $f(x)$ and $g(x)$ is defined as $f\Big(g(x)\Big)$. For example, if $f(x) = 3\,x^2 + 5$ and $g(x) = x - 4$, then

$$f\Big(g(x)\Big) \;=\; f(\,x-4\,) \;=\; 3\,(x-4)^2 + 5 \;=\; 3\,x^2 - 24\,x + 53\,.$$

We also use the notation $f \circ g\,(x)$ for the composition of f and g. It should be noted that in general, $f \circ g \neq g \circ f$. In the above example $g \circ f(x)$ is

$$g\Big(f(x)\Big) \;=\; g(\,3\,x^2+5\,) \;=\; 3\,x^2 + 5 - 4 \;=\; 3\,x^2 + 1$$

which is much different from $f \circ g(x)$.

The Chain Rule states that if g is differentiable at a value x and f is differentiable at the value $y = g(x)$ then the composition $f \circ g$ is differentiable at x with the derivative

$$(f \circ g)'\,(x) \;=\; f'(g(x)) \cdot g'(x)\,.$$

If we set $z = f(y)$ then the Chain Rule is easy to remember in the alternate (Liebnitz) notation as

$$\frac{dz}{dx} \;=\; \frac{dz}{dy} \cdot \frac{dy}{dx}\,.$$

For example, suppose we wish to differentiate $h(x) = (2\,x+5)^2$. We could either multiply it out and differentiate, or we could use the Chain Rule. For the latter technique, set $g(x) = 2\,x + 5$. Then $f(x)$ has to be x^2 to make $h(x) = f\Big(g(x)\Big)$. We have

$$h'(x) \;=\; f'\Big(g(x)\Big) \cdot g'(x)\,.$$

Now $f'(x) = 2\,x$, so $f'\Big(g(x)\Big) = 2\,g(x)$. Also $g'(x) = 2$. Then

$$h'(x) \;=\; 2\,g(x)\,2 \;=\; 4\,g(x) \;=\; 4(2\,x+5) \;=\; 8\,x + 20\,.$$

This result is easily verified by multiplying out $h(x)$ and differentiating directly.

In general, the derivative of $h(x) = (g(x))^n$ can be evaluated by the Chain Rule in the case where n is a positive integer. Set $f(x) = x^n$. Then $h(x) = (f \circ g)(x)$. Now $f'(x) = n\, x^{n-1}$. Thus

$$h'(x) \;=\; (f \circ g)'(x) \;=\; f'\Big(g(x)\Big) \cdot g'(x) \;=\; n\, g(x)^{n-1} \cdot g'(x)\,.$$

Given $g(x)$ we need only compute $g'(x)$ to find the derivative of $g^n(x)$.

A particularly useful example is the derivative of $g(x) = x^{n/m}$ where n is a nonzero integer, and m is a positive integer. We note that $g^m(x) = x^n$ and differentiate on both sides. Applying the Chain Rule to the left side with $f(x) = x^m$ and $h(x) = f\Big(g(x)\Big) = g^m(x) = x^n$, we find

$$\begin{aligned} h'(x) &= x^n \\ f'(g(x)) \cdot g'(x) &= n\, x^{n-1} \\ m\,(g(x))^{m-1} \cdot g'(x) &= n\, x^{n-1} \\ g'(x) &= \frac{n\, x^{n-1}}{m\, g^{m-1}(x)} \;=\; \frac{n\, x^{n-1}}{m\, x^{\frac{n}{m}(m-1)}} \;=\; \frac{n}{m}\, x^{n-1-\frac{n}{m}(m-1)} \\ &= \frac{n}{m}\, x^{\frac{n}{m}-1}\,. \end{aligned}$$

Exercises.

1. Let $f(x) = \dfrac{x+1}{x-1}$ and $g(x) = \dfrac{x-1}{x+1}$. Find $(f \circ f)(x)$ and $(f \circ g)(x)$ on the domain where each of these functions is defined.

Solution. We must use some algebra.

$$\begin{aligned} (f \circ f)(x) \;=\; f\,(f(x)) \;=\; \frac{f(x)+1}{f(x)-1} \;&=\; \frac{\dfrac{x+1}{x-1}+1}{\dfrac{x+1}{x-1}-1} \\ &= \frac{\dfrac{2x}{x-1}}{\dfrac{2}{x-1}} \;=\; \frac{2\,x}{2} \;=\; x\,. \end{aligned}$$

For $f \circ g$ the calculation is as follows.

$$\begin{aligned} (f \circ g)(x) \;=\; f\,(g(x)) \;=\; \frac{g(x)+1}{g(x)-1} \;&=\; \frac{\dfrac{x-1}{x+1}+1}{\dfrac{x-1}{x+1}-1} \\ &= \frac{\dfrac{2x}{x+1}}{\dfrac{-2}{x+1}} \;=\; \frac{2\,x}{-2} \;=\; -x\,, \end{aligned}$$

which is the answer in this case. ■

2. Use the Chain Rule to find $h'(x)$ for $h(x) = \sqrt{2\,x+1}$.

Solution. Set $f(x) = \sqrt{x}$ and $g(x) = 2\,x+1$. Then $f'(x) = \frac{1}{2}x^{-1/2}$ and $g'(x) = 2$. Since $h(x) = f\Big(g(x)\Big)$ we have

$$h'(x) \;=\; f'(g(x))\cdot g'(x) \;=\; \frac{1}{2}(2\,x+1)^{-1/2}\cdot 2 \;=\; (2\,x+1)^{-1/2}.$$

The answer is $\dfrac{1}{\sqrt{2\,x+1}}$. ■

3. Use the Chain Rule to find $h'(x)$ for $h(x) = (5x^2+3x+1)^2$.

Solution. Set $g(x) = 5\,x^2+3\,x+1$ and $f(x) = x^2$. Then $h(x) = f\Big(g(x)\Big)$ and $g'(x) = 10\,x+3$ while $f'(x) = 2\,x$. So $h'(x) = f'(g(x))\cdot g'(x) = 2\,(5\,x^2+3\,x+1)\cdot(10\,x+3)$ which simplifies to $100\,x^3+90\,x^2+38\,x+6$. ■

4. Use the Chain Rule to find $h'(x)$ for $h(x) = (x^3-x)^4$.

5. Use the Chain Rule to find $h'(x)$ for $h(x) = \sqrt{x^3+2x+1}$.

6. Use the Chain Rule to find $h'(x)$ for $h(x) = \left(x+\dfrac{1}{x}\right)^5$.

7. Using the definition of the derivative we have previously shown that

$$\frac{d}{dx}x^{1/2} \;=\; \frac{d}{dx}\sqrt{x} \;=\; \frac{1}{2}x^{-1/2}.$$

Use this fact and the Chain Rule to find the derivative of $h(x) = 7x^{3/2}-2x^{1/2}+3x^{-1/2}$.

Solution. We may set $g(x) = \sqrt{x}$ and choose $f(x) = 7x^3-2x+3/x$. Then $h(x) = f\Big(g(x)\Big)$ and $f'(x) = 21x^2-2-3/x^2$. Thus

$$\begin{aligned} h'(x) \;&=\; f'(g(x))\cdot g'(x) \;=\; \left(21(\sqrt{x})^2-2-\frac{3}{\sqrt{x}^2}\right)\cdot\frac{1}{2\sqrt{x}} \\ &=\; \left(21\,x-2-\frac{3}{x}\right)\cdot\frac{1}{2\sqrt{x}} \\ &=\; \frac{21\,x^2-2\,x-3}{2\,x^{3/2}}, \end{aligned}$$

which is the answer. ■

8. Using the definition of the derivative we have previously shown that

$$\frac{d}{dx}x^{1/3} \;=\; \frac{1}{3}x^{-2/3}.$$

Use this fact and the Chain Rule to find the derivative of $h(x) = x^{4/3} + x^{2/3} + 3x^{1/3}$.

9. Let $f(x) = \dfrac{ax+b}{x-1}$ where a and b are constants. Find all values of a and b such that $(f \circ f)(x) = x$ for all x in the domain of the composition function.

10. Use the Chain Rule to find $h'(x)$ for $h(x) = \sqrt{x^3-1} + \dfrac{1}{\sqrt{x^3-1}}$.