Magooey's Math Problems

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The Cauchy-Schwarz Inequality

Synopsis. The Cauchy-Schwarz Inequality is a general statement for real numbers that has a geometric interpretation. For $n = 2$, this inequality states that

$$|a_1 b_1 + a_2 b_2| \le \left(a_1^2 + a_2^2\right)^{1/2} \left(b_1^2 + b_2^2\right)^{1/2} .$$

For $n = 3$ the result is

$$|a_1 b_1 + a_2 b_2 + a_3 b_3| \le \left(a_1^2 + a_2^2 + a_3^2\right)^{1/2} \left(b_1^2 + b_2^2 + b_3^2\right)^{1/2} .$$

The right side clearly deals with the length of two line segments in two or three dimensional space. In general we have the following Theorem.

Theorem (Cauchy-Schwarz Inequality). *For real numbers a_1, a_2, ..., a_n and b_1, b_2, ... b_n the following holds.*

$$|a_1 b_1 + a_2 b_2 + \cdots + a_n b_n| \le \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)^{1/2} \left(b_1^2 + b_2^2 + \cdots + b_n^2\right)^{1/2} .$$

Proof. We begin an induction proof by considering the case $n = 2$. It is simply necessary to show, by squaring, that

$$(a_1 b_1 + a_2 b_2)^2 \le \left(a_1^2 + a_2^2\right) \left(b_1^2 + b_2^2\right) .$$

This is equivalent to

$$a_1^2 b_1^2 + 2\, a_1 a_2 b_1 b_2 + a_2^2 b_2^2 \le a_1^2 b_1^2 + a_1^2 b_2^2 + a_2^2 b_1^2 + a_2^2 b_2^2 ,$$

which can be rearranged to the following form.

$$\begin{aligned} 0 &\le a_1^2 b_2^2 - 2\, a_1 a_2 b_1 b_2 + a_2^2 b_1^2 \\ 0 &\le (a_1 b_2 - a_2 b_1)^2. \end{aligned}$$

The last is immediately obvious, so the case $n = 2$ has been verified.

Suppose we know that the Cauchy-Schwarz Inequality is true for all integers up to $n-1$ and that we wish to show the statement true for n. In particular we are given

$$|a_1b_1 + a_2b_2 + \cdots + a_{n-1}b_{n-1}| \;\le\; \left(a_1^2 + a_2^2 + \cdots + a_{n-1}^2\right)^{1/2} \left(b_1^2 + b_2^2 + \cdots + b_{n-1}^2\right)^{1/2} .$$

Set

$$\begin{aligned} A_{n-1} &= a_1^2 + a_2^2 + \cdots + a_{n-1}^2 \\ B_{n-1} &= b_1^2 + b_2^2 + \cdots + b_{n-1}^2 \\ C_{n-1} &= a_1b_1 + a_2b_2 + \cdots + a_{n-1}b_{n-1} \end{aligned}$$

.

By inductive hypothesis, we are given $|C_{n-1}| \le \sqrt{A_{n-1}B_{n-1}}$. We wish to show that $|C_{n-1} + a_nb_n| \le \sqrt{(A_{n-1} + a_n^2)(B_{n-1} + b_n^2)}$. By squaring, it is sufficient to show that

$$(C_{n-1} + a_nb_n)^2 \le \left(A_{n-1} + a_n^2\right)\left(B_{n-1} + b_n^2\right) .$$

Expanding, this is equivalent to

$$\begin{aligned} C_{n-1}^2 + 2a_nb_nC_{n-1} + a_n^2b_n^2 &\le A_{n-1}B_{n-1} + A_{n-1}b_n^2 + a_n^2B_{n-1} + a_n^2b_n^2 \\ C_{n-1}^2 + 2a_nb_nC_{n-1} &\le A_{n-1}B_{n-1} + A_{n-1}b_n^2 + a_n^2B_{n-1} \end{aligned}$$

By hypothesis, we have $C_{n-1}^2 \le A_{n-1}B_{n-1}$. Therefore it is sufficient to prove

$$2a_nb_nC_{n-1} \;\le\; A_{n-1}b_n^2 + a_n^2B_{n-1} .$$

We recall the elementary inequality

$$\sqrt{xy} \;\le\; \frac{x+y}{2}, \qquad x,\, y \ge 0$$

which is simple to prove (and is in fact the case $n = 2$ of the Theorem on Arithmetic Means and Geometric Means). Applying this with $x = A_{n-1}b_n^2$ and $y = B_{n-1}a_n^2$ we find that

$$\begin{aligned} \sqrt{A_{n-1}b_n^2 \cdot B_{n-1}a_n^2} &\le \frac{A_{n-1}b_n^2 + B_{n-1}a_n^2}{2} \\ 2\,|a_nb_n|\,\sqrt{A_{n-1}B_{n-1}} &\le A_{n-1}b_n^2 + B_{n-1}a_n^2 \end{aligned}$$

However, by inductive hypothesis and some algebra, we have

$$2a_nb_nC_{n-1} \;\le\; 2\,|a_nb_n|\,C_{n-1} \;\le\; 2\,|a_nb_n|\,\sqrt{A_{n-1}B_{n-1}}$$

and the last part we just showed was less than or equal to $A_{n-1}b_n^2 + B_{n-1}a_n^2$. Therefore the Cauchy-Schwarz Inequality for the case of n follows, and the Inequality is proved for all positive integers n. ■